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VICTOR P. STARR, DIRECTOR

10 JULY 1963

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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
BEDFORD, MASSACHUSETTS

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ENERGETICS OF THE SOLAR SEMIDIURNAL TIDE IN THE ATMOSPHERE

BY

WALTER L. JONES

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ENERGETICS OF THE SOLAR SEMIDIURNAL TIDE IN THE ATMOSPHERE

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Walter L. Jones

ABSTRACT

A study is made of the energy generation, transport, and dissipation by the solar semidiurnal tide in the earth's atmosphere. Computations based on recent observations at Terciera, Azores show a downward transport of available potential energy in the troposphere, reaching a maximum of 7×10^{-3} watts per square meter at or near the ground. Similar data for Fort Worth, Texas, substantiates the assumptions used in this calculation. This flux is generated primarily by water vapor insolational heating, though horizontal convergence of tidal available potential energy may be significant.

Neither eddy viscosity nor an inverse correlation between convective heating and tidal temperature fluctuation appear adequate as energy sinks for this flux. Instead, it is proposed that the undulations of the earth's surface interact with the main tidal motion to generate secondary internal gravity waves; these propagate energy vertically to levels where they are viscously damped, and thus represent a loss of energy to the tide.

The complex form of the tidal-terrain interaction prohibits a rigorous computation of its magnitude. Two very approximate approaches yield energy fluxes of 0.5×10^{-3} and 30×10^{-3} watts per square meter, respectively; both figures are of an order-of-magnitude nature. Thus no definite conclusions can be reached about the importance of its effect. If the effect is not the source of the downward energy flux, the effects of eddy viscosity or convective heating must be greater than computed, or some unknown energy sink must exist.

Computations are made of the extent to which eddy and molecular viscosity damp the secondary internal gravity waves; waves longer than 3000 kilometers are damped in the thermosphere, by molecular viscosity and possible by hydromagnetic damping. Waves from 1000-3000 kilometers in length are damped in the mesosphere, and waves of length 200-1000 kilometers are damped in the troposphere. Still shorter waves must be treated in a fully viscous theory.

The Terciera and Fort Worth data show a northward meridional transport of tidal energy. Both stations also show meridional transports of zonal momentum by the semidiurnal tide. A similar transport, with a seasonal fluctuation, is found in meteor trail observations at 92 km. Theoretical aspects of these transports and their relation to advection of sensible heat and potential energy are discussed.

Finally, a simplified approach to non-linear tidal theory is taken to show that interaction between tidal and Rossby waves cannot account for the observed energy loss.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

CHAPTER I. INTRODUCTION

A. Observed Atmospheric Tides

Three excellent review articles are available to the serious student of atmospheric tides. These are the chapter, Atmospheric Tides and Oscillations, by Chapman, (1950) in the Compendium of Meteorology; the book, Oscillations of the Earth's Atmosphere, by Wilkes, (1949); and the Chapter, Atmospheric Tides, in Advances in Geophysics, Vol. 7, by Siebert, (1961). The last is especially recommended as being most recent and most complete in its theoretical development. The first two offer more extensive accounts of observational data. As these references are generally available, only a limited outline of the observed tides and the history of tidal theory will be given here.

As early as 1727, Newton noted that universal gravitation implied an atmospheric tide. He correctly surmised that it would be a very small effect. By the latter part of the eighteenth century, the solar semidiurnal tide had been observed and was known to such mathematicians as Bernoulli and D'Alembert, (Wilkes, 1949). The first complete dynamical theory of tides was developed by Laplace, (1799, 1825).

Laplace also initiated the search for the lunar semidiurnal tide in the atmosphere. The major incongruity of the atmospheric tides is that the solar semidiurnal tide is thirty or forty times as large as its lunar counterpart, while in the ocean, the lunar tide is dominant. In fact, the lunar tide was too small for Laplace to recover from eight years of

ground pressure data at Paris. Several others made unsuccessful attempts in the next two decades to find the lunar tide.

The first reliable determination of the lunar tide was made by Le Froy at St. Helena, in 1842, (Sabine, 1847). It was relatively more simple for him to obtain tidal data, both because of the increased magnitude of the lunar tide at low latitudes, and because of the relative constancy of barometric readings in the tropics.

Other observations, especially of the solar tide, and by the end of the nineteenth century, extensive compilations of the 24-, 12-, and 8-hour tides were available, (Hann, 1889, 1889, 1918). Praminik, (1926) computed similar data for the 6-hour tide. As early as 1890, Schmidt noted that the distribution of the phase of the semidiurnal tide could be explained by assuming two modes of oscillation, one progressing in a westward direction, the other zonal and oscillating with Greenwich, rather than local time.

Simpson, (1918), extended Schmidt's idea, applying it to the data of 214 stations. He found two twelve-hour waves, given by the empirical formulae:

$$S_2^2 = 1.25 \cos^3\theta \sin[\sigma t + 2\phi + 154^\circ] \quad \text{mb}$$
$$S_2^0 = 0.061(3\sin^2\theta - 1)\sin[\sigma t + 105^\circ] \quad \text{mb} \tag{1}$$

Here t is Greenwich time, σ the tidal frequency, θ the latitude, and ϕ the longitude.

Haurwitz, (1956) extended the analysis of the twelve-hour tide to 296 stations, and Kertz, (1959), analyzed Haurwitz data in terms of spherical harmonic functions. The major components of the semi-diurnal tide are:

$$
\begin{aligned}
S_2^2 &= [1.23\,P_2^2(\theta) - 0.224\,P_2^2(\theta)]\sin(\sigma t + 2\phi + 168^\circ)\ \text{mb.} \\
S_2^0 &= [7.18\,P_2^0(\theta)\sin(\sigma t + 135^\circ) + 5.62\,P_3^0(\theta)\sin(\sigma t + 125^\circ)] \\
&\quad \times 10^{-2}\ \text{mb.} \qquad (2) \\
S_2^3 &= 10.70\,P_3^3(\theta)\sin(\sigma t + 3\phi + 88^\circ) \times 10^{-2}\ \text{mb.}
\end{aligned}
$$

The diurnal tide contrasts with the semidiurnal tide, inthat it is quite variable, both in a random sense, and as a function of season. It is also quite dependent on the height of the station above sea level. It is generally of the same order as or smaller than the semidiurnal tide at the surface of the earth.

The eight- and six-hour tidal components are a few tenths the size of the semidiurnal tide, and are also quite irregular. It is the predominant semidiurnal tide which has attracted the most attention, and which will be the principal topic of this dissertation.

B. History of the Resonance Theory of Tides

As was noted, the first dynamic theory of the tides was developed by Laplace, (1799, 1825). Although most of his studies dealt with incompressible oceans of uniform depth, he showed the theory could be applied to a compressible atmosphere, providing three assumptions are made:

1. Vertical accelerations are negligible.
2. The atmosphere is isothermal.
3. Oscillations occur under isothermal conditions.

The first assumption continues to be used widely in tidal theory, as well as in many other atmospheric problems. The second is an oversimplification that obscures effects which later researchers considered vital. Nevertheless, it is a great convenience, when used judiciously, and can be used to illustrate important points at least qualitatively. The third assumption does not appear valid at all. First, external heating may be important, and second, even in the absence of heating, tidal motions would be more nearly adiabatic than isothermal. Chapman, (1932) has shown this to be true for the lunar semidiurnal tide, for example.

Laplace also noted the disparity in magnitude between solar and lunar tides. He attributed this to a heating source for the solar tide, of much greater magnitude than the gravitational source of the lunar tide.

Kelvin, (1882), pointed out that if this were the case, one would expect the solar diurnal tide to be substantially larger than the semidiurnal tide, since the diurnal heating is larger than the semidiurnal. As a solution to this dilemma, he suggested that the atmosphere has a natural resonance near twelve hours. The isothermal atmosphere of Laplace possessed such a resonance, whose frequency was determined by the atmospheric temperature. Kelvin's suggestion led to a half-century and more of work on the so-called "resonance theory" of tidal motions by many authors.

Rayleigh, (1890), and Margules, (1892), were the first to investigate the periods of free oscillations in search for the resonance. Rayleigh made several simplifications, among them the critical neglect of the earth's rotation. Margules included the rotation, but based his work on Laplace's theory, and so suffered from the same assumptions and their limitations. Margules, (1890), also considered the oscillations of an atmosphere subject to periodic heating.

Lamb, (1910, 1916), extended Laplace's theory to an atmosphere in convective equilibrium, with an adiabatic lapse rate and adiabatic changes of state. This model gave results very similar to that of Laplace. Both behaved like an incompressible fluid with a depth equal to the scale height at the base of the atmosphere, and so possessed a natural resonant frequency. Lamb also showed that an atmosphere with a uniform but non-adiabatic lapse rate has an infinite number of resonances of different frequencies. Taylor, (1936), extended Lamb's model to a rotating sphere. The possibility of multiple resonances was later to be vital to the resonance theory.

In 1924 Chapman considered eddy conduction of heat from the surface of the earth as a tidal driving force. The phase for such a thermally driven tide was nearly in quadrature with that of a gravitationally induced tide; the observed phase of the semi-diurnal progressive tide lay about midway between the two. From this observation, as well as from computed values of eddy heat transport, Chapman concluded that the thermal and gravitational driving forces were nearly equal in their effectiveness. For some time, it was thought that the phase of the semidiurnal tide had been explained in this manner. It was still necessary to invoke a strong

resonance to explain the magnitude of the tide.

Bartels, (1927), then developed a two-layer model, with a troposphere of constant lapse rate, and an isothermal stratosphere. This model had a single free wave eigensolution, with a period of about 10.5 hours. This was not close enough to the twelve hour tide to produce a large amplification, however. Taylor, (1929) showed that this resonance did agree well with the velocity of the Krakatoa pressure wave of 1883. For a short while, this model posed a serious problem to adherents of the resonance theory. However, Taylor himself, (1936), eliminated the difficulty by pointing out the possibility of multiple resonances.

In 1937 Pekaris developed a five-layer model, having a 220 K. isothermal stratosphere, a 190 K. isothermal top, and a region of temperature maximum, around 350 K., just below 60 km. This temperature profile was in good accord with what was then known of the upper atmosphere: The temperature maximum had been inferred from anomalous sound propagation, (Whipple, 1918), and the low temperature above 80 km. from the presence of noctilucent clouds.

The Pekaris model had two resonant frequencies, one at 10.5 hours and a second at 12.0 hours. This theory thus appeared to explain both the Krakatoa wave speed and the resonance of the semidiurnal tide. When combined with the explanation by Chapman of the phase, the semidiurnal tide seemed well explained.

In the last decade, however, several criticisms of the resonance theory have been raised. Siebert, (1957), and Kertz, (1959), have observed

other wave types of 6- and 8- hour frequencies. While interpretation of their analyses is difficult, it appears that these waves would need resonances amplification comparable to that of the semidiurnal wave; no such resonances have been proposed theoretically.

More damaging are the recently obtained temperature data for the temperature maximum. These data, mostly from rocket soundings, show a much lower temperatuere, less than 300 K. The ARDC 1959 model atmosphere, for example, shows a maximum of 283 K., (Minzner, Champion, and Pond, 1959). Jacchia and Kopal, (1952), investigated the sensitivity of the Pekaris model to changes in the temperature profile. It is quite sensitive to changes in the temperature maximum; a change of ten or fifteen degrees alters the semi-diurnal resonance amplification very markedly, and a reduction to 300 K. causes it to disappear entirely, leaving only the tropospheric mode corres-ponding to the Bartels model.

Finally, the Pekaris model predicts a mode in the pressure wave near 30 km. Recent data by Harris, Finger, and Teweles, (1962), for the semi-diurnal tide shows no such node up to a height of 10 mb., or 31 km. It is possible such a node exists at a greater height, but if so, there is no trace of it at this level.

C. Recent Contributions to Tidal Theory

Without the aid of a strong resonance amplification, gravitational and eddy-heating generating forces are inadequate to explain the observed tidal amplitudes. Recently, attention has been turned to insolational

heating by water vapor, carbon dioxide, and ozone absorption.

Sen and White, (1955), and White, (1956) developed an improved tidal theory, taking into account heating of any form and at any height. Independently, Siebert, (1955), derived a similar theory; as it is somewhat simpler and follows more directly along the main line of development of tidal theory, Siebert's theory will be taken as the basis for the following work.

Siebert (1961) made use of his theory to analyze insolational heating by water vapor. He used a rather artificial atmospheric model, neither isothermal, nor adiabatic, but representative of a troposphere with a gradual transition into a stratosphere. This model has one mode of oscillation, close to that of Bartels. Siebert made use of the empirical Mügge - Möller, (1932) equation for water vapor absorption. The model showed a resonance amplification of 3.7, and developed a tide one-third the magnitude of the observed tide.

Thus, insolational heating by water vapor appears considerably more important than either eddy conductivity or gravitational potential as a source of the semidiurnal tide. It is, at least according to Siebert, not adequate to explain the observed tide. He proposes three possibilities:

1. There is a resonance of a form that has been overlooked.
2. The insolational heating coefficients are too small.
3. There are sources that have not been discovered.

As one additional source, Siebert considered ozone absorption, concluding that it is perhaps a third as effective as water vapor, but

recognizing the approximate nature of both the model and the absorption data. Small and Butler, (1961), also considered ozone heating, using a more appropriate temperature profile, (Murgatroyd, 1957). They concluded that ozone heating is capable of producing both the observed amplitude and phase of the semidiurnal tide, and felt the unrealistic form of Siebert's temperature profile led to his smaller amplitudes for ozone heating.

In recent years, some knowledge of tidal oscillations as a function of height has been obtained. Meteor trails have been observed by radar, and the tidal winds deduced at 80 to 100 km., (Greenhow and Neufeld, 1955, 1956, Elford, 1953). One series of 23 hourly rocket soundings has been analyzed for tidal winds in the height range of 35 to 65 kilometers, (Lenhard, 1963). At lower levels, Wagner, (1932), and Stapf, (1934), observed the tides at several Alpine stations, and found a phase lag with height. This has been confirmed by radiosonde observations reduced by Harris, Finger, and Teweles, (1962). J. Bjerknes (1949) has qualitatively interpreted this phase lag as the result of ground friction. Since Chapman's explanation of the tidal phase was based on mechanisms requiring resonance, it no longer is valid, and some such explanation of phase shift is a necessity.

D. Scope of the Present Study

This study is primarily concerned with the generation of secondary internal gravity waves with tidal frequency, but shorter wavelength. The primary source of these waves is a tidal-terrain interaction, though non-linear interaction with the large scale atmospheric eddies is also considered.

The secondary waves propagate energy vertically to levels where they are damped, by eddy viscosity, molecular viscosity, or possibly magnetic damping. As these waves extract energy from the primary tide, they represent a loss mechanism for the latter, exerting a substantial influence on its character. In particular, the mechanism offers an explanation of the pressure phase at the ground.

Chapter II discusses the traditional linearized and inviscid tidal equations, essentially as developed by Pekaris, (1936), Weekes and Wilkes, (1947), Wilkes, (1949), and Siebert, (1955, 1961). Non-adiabatic heating is included, and rotating planar and spherical geometries are treated.

Chapter III discusses the boundary conditions that must be imposed on the vertical wave equation. In particular, it is shown that the normal assumption of no vertical velocity of the primary wave at the ground may not be valid. If the surface is undulatory, the horizontal motions would transfer mass through the terrain slope unless there are additional waves to counteract this transfer. These waves are related in scale to the terrain disturbances, and have tidal period. A Fourier analysis of the earth's surface around a latitude circle is used with a planar model to obtain an estimate of the vertical energy transport by the secondary waves, and hence of the downward energy transport required in the primary tidal wave. A very approximate analysis, in all probability an underestimate, gives a vertical energy flux $\sim 0.5 \times 10^{-3}$ watts per square meter.

In Chapter IV, observational data for the semidiurnal tide above the Azores, (Harris, Finger, and Teweles, 1962), are considered. Under

the logical and self-consistent, though yet unproven assumption that one known mode of oscillation prevails throughout the troposphere and lower stratosphere at that location, it is found that there is a downward flux of tidal energy. This flux is at a maximum at the surface, where secondary waves are generated; it requires the generation of 7×10^{-3} watts per square meter of tidal energy, by a source located in the troposphere. Insolational heating by water vapor seems to satisfy this requirement. This flux is an order of magnitude greater than the lower limit predicted for the tidal-terrain effect; unfortunately, an upper limit has not been established.

As the theory of secondary waves developed in Chapter III depends on the assumption that they are dissipated at some level in the atmosphere, the effects of eddy and molecular viscosity are taken up in Chapter V. Data on these viscosities are meager and subject to problems of interpretation, but it appears that the bulk of the secondary waves are dissipated in the troposphere. The longest waves transport 10 or 20% of the energy to the mesosphere or even the lower ionosphere, where molecular viscosity becomes important. Eddy viscosity does not appear to play an important role in the primary semidiurnal tide, as it is observed in the troposphere. It is concluded from theory and observation that viscous losses do not account for the downward flux of tidal energy.

Another possible source of dissipation, hydromagnetic damping in the ionosphere, is considered in Chapter VI. The assumptions are crude, but provide an upper limit to damping, which is comparable to molecular

viscosity for waves of interest in the E-layer.

A third possible sink for the downward flux of tidal energy can be found in convection of heat from the ground. Just as a positive correlation between temperature and insolational heating brings about a generation of tidal energy, so a negative correlation between convective heating and density might provide for its dissipation. This possibility is examined in Chapter VII. From the limited knowledge of convective heating, it appears too small, by less than an order of magnitude. It is also argued that such an effect should produce abrupt changes in tidal pressure between the ground and 900 mb levels that are not observed.

Chapter VIII considers other aspects of the observed tide, including meridional transport of angular momentum and tidal energy, and the generation of tidal kinetic energy. The equations of motion are used to show how these quantities, all zero for a single linear mode, may be non-zero in the presence of a mixture of modes of oscillation.

Non-linear interaction between tidal and Rossby waves are treated in Chapter IX. A wave equation analogous to the linear case is derived for the case where the mean magnitude of all waves is constant. This equation is applied to an elementary model in order to obtain an order of magnitude estimate of the energy transport by secondary waves so generated. Little energy is lost in long wavelengths; it is possible, but not likely that short waves provide a considerable loss, but most of their energy is derived from the Rossby waves.

CHAPTER II. THE LINEAR TIDAL EQUATIONS

A. Major Assumptions

The following linearized tidal theory is, in large part, that presented by Siebert, (1961). His work in turn follows directly from those of Pekaris, (1937), and Wilkes, (1949). Siebert's nomenclature will be retained, except as noted. In subsequent chapters, both rotating planar and rotating spherical geometries will be used, and the tidal equations will be derived for both cases.

Three major assumptions, all traditional to tidal theory, will be made at the outset. The first is the assumption of negligible vertical accelerations, the hydrostatic approximation used in much of meteorology. Eckart, (1960), has developed atmospheric wave equations along somewhat different lines without this assumption. The effect of the approximation is to change terms of the form $N^2 - \sigma^2$ to the form N^2. N is the Väisälä-Brunt frequency, and σ the tidal frequency. As the former corresponds to periods of a few minutes, while the latter corresponds to periods of several hours, the error introduced by the hydrostatic approximation is quite small.

The second assumption is that all terms involving the horizontal component of the earth's angular velocity are negligible. These terms appear as vertical accelerating forces or in conjunction with small vertical velocities, and are also normally neglected in meteorological work. Again, Eckart has considered this approximation in some detail. In the case of

the rotating plane, the equations may be solved, albeit with more difficulty. The error introduced by the assumption is small unless the tidal angular frequency is nearly twice the vertical angular velocity of the plane, crudely corresponding to the poles for the semidiurnal tide, and a latitude of thirty degrees for the diurnal tide.

In the case of a rotating sphere, the separation of horizontal and vertical variation of parameters is no longer possible unless this approximation is made, and no solution to the tidal equations is known for this case. Eckart accepts this approximation with reservations, therefore.

Finally, it will be assumed that the undisturbed atmosphere has only vertical variation of such parameters as pressure or temperature, or at least that horizontal variations contribute only terms small enough to be dropped in the linearization of equations.

B. The Rotating Plane

In this model, the earth is assumed to be planar and smooth, with a vertically directed angular velocity, $\vec{\omega}$. The gravitational acceleration, $\vec{g}$ is assumed to be constant and everywhere uniform. The atmosphere is taken to be inviscid and of uniform composition. Atmospheric parameters such as temperature, T , pressure, P , and density, ρ , are assumed to consist of an undisturbed, time-invariant component and a tidal component. The former will be denoted by the subscript $_o$, and the latter by the subscript $_r$, or $_n$ if they belong to a specific mode of oscillation.

Thus:

$$T = T_0 + T_\sigma = T_0 + \sum_n T_n \tag{1}$$

for example. The zero-order hydrostatic equation is:

$$\frac{dP_0}{dz} = -g\rho_0 \tag{2}$$

and the equation for an ideal gas is:

$$P_0 = \frac{R}{M}\rho_0 T_0 = g\rho_0 H_0 \tag{3}$$

where R is the universal gas constant, M the mean molecular weight of the air, z the vertical coordinate, and H_0 the atmospheric scale height.

The equation of motion is:

$$\frac{D\vec{V}_\sigma}{Dt} + 2\vec{\omega}\times\vec{V}_\sigma = -\frac{1}{\rho}\,\mathrm{grad}\,P + \vec{g} \tag{4}$$

As the tidal potential is assumed to have negligible importance for the solar tides, it is omitted from equation (4). (It is included in Siebert's work). If (4) is linearized on the assumptions that perturbations are small compared to undisturbed quantities and that velocities are small compared to $a\omega$, where a is a tidal scale distance:

$$\frac{\partial\vec{V}_\sigma}{\partial t} + 2\vec{\omega}\times\vec{V}_\sigma \simeq -\frac{1}{\rho_0}\,\mathrm{grad}\,P_0 + \frac{\rho_\sigma}{\rho_0}\vec{g} \tag{5}$$

The horizontal coordinates of the plane are taken to be ξ and η, and the corresponding velocities as u and v. The vertical

velocity is w . If vertical accelerations are ignored, (5) may be written as:

$$\frac{\partial u_\sigma}{\partial t} - 2\omega v_\sigma = -\frac{1}{\rho_0}\frac{\partial p_\sigma}{\partial \xi} \tag{6}$$

$$\frac{\partial v_\sigma}{\partial t} + 2\omega u_\sigma = -\frac{1}{\rho_0}\frac{\partial p_\sigma}{\partial \eta} \tag{7}$$

$$\frac{\partial p_\sigma}{\partial z} = -g\rho_0 \tag{8}$$

The equation of continuity is:

$$\frac{D\rho}{Dt} + \rho_0 \chi_\sigma = 0 \tag{9}$$

where the velocity divergence is written as:

$$\chi_\sigma \equiv \frac{\partial u_\sigma}{\partial \xi} + \frac{\partial v_\sigma}{\partial \eta} + \frac{\partial w_\sigma}{\partial z} \tag{10}$$

The first law of thermodynamics may be written as:

$$\delta Q = C_v\, dT + P\, d\left(\frac{1}{\rho}\right) \tag{11}$$

δQ is an infinitesimal amount of heat added per unit mass, and C_v and C_p are the specific heats of air at constant volume and pressure, respectively. Both specific heats are assumed to be constant and spacially uniform. If there is a periodic addition of heat from external sources, which may be

described by a heating rate J_σ units of heat per unit time per unit mass, then:

$$\delta Q = J_\sigma \, dt \tag{12}$$

If (12) is substituted into (11), and use is made of the relations:

$$R = M(C_p - C_v) \tag{13}$$

$$\gamma \equiv C_p / C_v \tag{14}$$

where M is the mean molecular weight of the air.

Then equation (11) becomes:

$$\frac{R}{M(\gamma - 1)} \frac{D T_\sigma}{D t} = \frac{P_0}{\rho_0^2} \frac{D \rho_\sigma}{D t} + J_\sigma \tag{15}$$

If equation (3) is differentiated, it becomes:

$$\frac{dP}{P} - \frac{d\rho}{\rho} = \frac{dT}{T} \tag{16}$$

This equation may be used to eliminate T from (15), with the result that:

$$\frac{D P_\sigma}{D t} = \gamma g H_0 \frac{D \rho_\sigma}{D t} + (\gamma - 1) \rho_0 J_\sigma \tag{17}$$

In its linearized form:

$$\frac{D P_\sigma}{D t} = \frac{\partial P_\sigma}{\partial t} + w_\sigma \frac{d P_0}{d z} \tag{18}$$

Similar results are obtained for the substantial derivatives of T and ρ.

These basic equations may now be used to develop a set of tidal equations. It is assumed that all tidal variables have time dependence of the form $e^{i\sigma t}$, so that:

$$\frac{\partial}{\partial t} = i\sigma \tag{19}$$

Equations (6) and (7) may be solved for u_σ and v_σ, with the aid of (19):

$$u_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{\rho_0}\frac{\partial p_\sigma}{\partial \xi} + \frac{2\omega}{\rho_0}\frac{\partial p_\sigma}{\partial \eta}\right] \tag{20}$$

$$v_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{\rho_0}\frac{\partial p_\sigma}{\partial \eta} - \frac{2\omega}{\rho_0}\frac{\partial p_\sigma}{\partial \xi}\right] \tag{21}$$

If a differential operator F is defined by:

$$F \equiv \frac{a}{(\sigma^2/4\omega^2 - 1)}\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right] \tag{22}$$

where a is a characteristic distance, then (20), (21), and (22) may be substituted into (10), obtaining:

$$\chi_\sigma = \frac{\partial w_\sigma}{\partial \zeta} + \frac{i\sigma}{4\omega^2 a^2} F\left(\frac{p_\sigma}{\rho_0}\right) \tag{23}$$

The objective will be to obtain from (23) an equation in χ_σ, which may be solved by the method of separation of variables. If equations (9), (17), and (18) are combined, and use is made of (19):

$$i\sigma P_\sigma + w_\sigma \frac{\partial \rho_0}{\partial z} = \gamma g H_0 \rho_0 X_\sigma + (\gamma-1)\rho_0 J_\sigma \tag{24}$$

or on application of (2):

$$i\sigma P_\sigma = W_\sigma g \rho_0 + \gamma g H_0 \rho_0 X_\sigma + (\gamma-1)\rho_0 J_\sigma \tag{25}$$

If this is in turn differentiated with respect to z :

$$i\sigma \frac{\partial P_\sigma}{\partial z} = g\rho_0 \frac{\partial W_\sigma}{\partial z} - \frac{g W_\sigma \rho_0}{H_0}\left(1 + \frac{\partial H_0}{\partial z}\right) + \gamma g \rho_0 X_\sigma \tag{26}$$

$$- \gamma g H_0 \rho_0 \frac{\partial X_\sigma}{\partial z} + (\gamma-1)\frac{\partial}{\partial z}(\rho_0 J_\sigma)$$

Here use has been made of the fact that:

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = -\frac{1}{H_0}\left(1 + \frac{\partial H_0}{\partial z}\right) \tag{27}$$

The equation of continuity and the hydrostatic equation may be combined to obtain:

$$i\sigma \frac{\partial P_\sigma}{\partial z} = -i\sigma g \rho_\sigma = W_\sigma g \frac{\partial \rho_\sigma}{\partial z} + g\rho_0 X_\sigma \tag{28}$$

Equations (26) and (28) may be combined to obtain:

$$\frac{\partial W_\sigma}{\partial z} = \gamma H_0 \frac{\partial X_\sigma}{\partial z} - (\gamma-1) X_\sigma - \frac{(\gamma-1)}{g\rho_0}\frac{\partial}{\partial z}(\rho_0 J_\sigma) \tag{29}$$

If this equation and (23) are differentiated with respect to ζ, and $\frac{1}{\gamma}\frac{\partial^2 W_\sigma}{\partial \zeta^2}$ eliminated between them, the resulting equation is:

$$\frac{1}{\gamma}\frac{\partial X_\sigma}{\partial \zeta} + \frac{g}{4\omega^2 a^2} F\left[-\frac{i\sigma}{\gamma g}\frac{\partial}{\partial \zeta}\left(\frac{P_\sigma}{p_0}\right)\right] \tag{30}$$
$$= H_0 \frac{\partial^2 X_\sigma}{\partial \zeta^2} + \frac{\partial H_0}{\partial \zeta}\frac{\partial X_\sigma}{\partial \zeta} - \kappa \frac{\partial X_\sigma}{\partial \zeta} - \frac{\kappa}{\rho_0 g}\frac{\partial^2}{\partial \zeta^2}(\rho_0 J_\sigma)$$
$$- \frac{\kappa}{\rho_0 g H_0}\left(1 + \frac{dH_0}{d\zeta}\right)\frac{\partial}{\partial \zeta}(\rho_0 J_\sigma)$$

where $\kappa = \frac{\gamma - 1}{\gamma}$. But from (25) and (29):

$$-\frac{i\sigma}{\gamma g}\frac{\partial}{\partial \zeta}\left(\frac{P_\sigma}{p_0}\right) = \left(\kappa + \frac{\partial H_0}{\partial \zeta}\right)X_\sigma - \frac{\kappa}{g}\left(1 + \frac{\partial H_0}{\partial \zeta}\right)J_\sigma \tag{31}$$

On substitution of (31) in (30) and further simplification:

$$H_0 \frac{\partial^2 X_\sigma}{\partial \zeta^2} + \left(\frac{\partial H_0}{\partial \zeta} - 1\right)\frac{\partial X_\sigma}{\partial \zeta} - \frac{\kappa}{g}\frac{\partial}{\partial \zeta}\left[\frac{\partial J_\sigma}{\partial \zeta} - \left(1 + \frac{\partial H_0}{\partial \zeta}\right)\frac{J_\sigma}{H_0}\right]$$
$$- \frac{g}{4\omega^2 a^2} F\left[\left(\kappa + \frac{\partial H_0}{\partial \zeta}\right)X_\sigma - \frac{\kappa}{g}\left(1 + \frac{\partial H_0}{\partial \zeta}\right)J_\sigma\right] \tag{32}$$
$$= 0$$

Equation (32) may be solved by the method of separation of variables. Let X_σ and J_σ be represented by expansions in the eigenfunctions of the operator F:

$$X_\sigma = \sum_n X_n(\zeta)\,\Psi_n(\mathfrak{J}, \eta) \tag{33}$$
$$J_\sigma = \sum_n J_n(\zeta)\,\Psi_n(\mathfrak{J}, \eta)$$

When these coefficients are substituted into (32), with h_n taken as the separation coefficient:

$$F\,\Psi_n + \frac{4a^2\omega^2}{g h_n}\,\Psi_n = 0 \tag{34}$$

and:

$$H_0 \frac{d^2 X_n}{d z^2} + \left(\frac{dH_0}{dz} - 1\right)\frac{dX_n}{dz} + \left(\kappa + \frac{dH_0}{dz}\right)\frac{X_n}{h_n} =$$

$$\frac{\kappa}{g}\left\{\left(1+\frac{dH_0}{dz}\right)\frac{J_n}{H_0 h_n} - \frac{d}{dz}\left[\left(1+\frac{dH_0}{dz}\right)\frac{J_n}{H_0}\right] + \frac{d^2 J_n}{dz^2}\right\} \tag{35}$$

The value of the separation coefficient is determined by the eigensolutions to equation (34). If J_n and H_0 are known functions of z, then equation (35) can in principal be solved for X_n. It is possible to express all other tidal parameters in terms of X_n.

It is conventional to transform equation (35) into the form of a one-dimensional wave equation by two further changes of variable. Let:

$$x \equiv \int_0^z \frac{dz'}{H_0(z')} \tag{36}$$

and:

$$y_n e^{x/2} \equiv X_n - \frac{\kappa J_n}{g H_0} \tag{37}$$

When (36) and (37) are substituted into (35), the resulting equation is:

$$\frac{d^2 y_n}{d x^2} - \frac{1}{4}\left[1 - \frac{4}{h_n}\left(\kappa H_0 - \frac{dH_0}{dx}\right)\right] y_n = \frac{\kappa J_n}{\gamma g h_n} \tag{38}$$

Eigensolutions for the pressure and velocities can also be expressed in terms of the wave function y_n, (Siebert, 1961):

$$P_n = \frac{\gamma h_n P_0(0) e^{-x/2}}{i\sigma\, H_0(x)} \left(\frac{dy_n}{dx} - \frac{1}{2} y_n\right) \tag{39}$$

$$w_n = \gamma h_n e^{x/2}\left[\frac{dy_n}{dx} + \left(\frac{H_0(x)}{h_n} - \frac{1}{2}\right) y_n\right] \tag{40}$$

$$u_n = \frac{\gamma g h_n e^{x/2}}{(\sigma^2 - 4\omega^2)} \left(\frac{dy_n}{dx} - \frac{1}{2} y_n\right)\left[\frac{\partial}{\partial \xi} - \frac{2\omega}{i\sigma}\frac{\partial}{\partial \eta}\right] \tag{41}$$

$$v_n = \frac{\gamma g h_n e^{x/2}}{(\sigma^2 - 4\omega^2)} \left(\frac{dy_n}{dx} - \frac{1}{2} y_n\right)\left[\frac{\partial}{\partial \eta} + \frac{2\omega}{i\sigma}\frac{\partial}{\partial \xi}\right] \tag{42}$$

C. The Spherical Model

In the case of this model, the earth will be taken to be a smooth sphere, with radius a. Latitude will be denoted by θ, and longitude by ϕ. The northward and eastward winds will correspondingly be denoted by v and u. (Note that Siebert uses θ as co-latitude, u as the

southward wind, and V as the eastward wind.) The radial distance r to a point at a height z, ($r = z + a$) will be replaced by a if it appears as a factor of another variable. All other quantities will remain as they were defined in the previous section.

In spherical coordinates, the divergence equation becomes:

$$\chi_\sigma = \frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}\left(V_\sigma\cos\theta\right) + \frac{1}{a\cos\theta}\frac{\partial u_\sigma}{\partial\phi} + \frac{\partial w_\sigma}{\partial z} \tag{43}$$

and the horizontal motion equations become:

$$\frac{\partial u_\sigma}{\partial x} - 2\omega\sin\theta\, V_\sigma = -\frac{1}{a\rho_0\cos\theta}\frac{\partial P_\sigma}{\partial\phi} \tag{44}$$

and

$$\frac{\partial V_\sigma}{\partial x} + 2\omega\sin\theta\, u_\sigma = -\frac{1}{a\rho_0}\frac{\partial P_\sigma}{\partial\theta} \tag{45}$$

These two equations may be solved for u_σ and V_σ, yielding:

$$u_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{a\rho_0\cos\theta}\frac{\partial P_\sigma}{\partial\phi} + \frac{2\omega}{a\rho_0}\frac{\partial P_\sigma}{\partial\theta}\right] \tag{46}$$

and:

$$V_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{a\rho_0}\frac{\partial P_\sigma}{\partial\theta} - \frac{2\omega}{a\rho_0\cos\theta}\frac{\partial P_\sigma}{\partial\phi}\right] \tag{47}$$

It is also feasible to redefine the differential operator F . If $f = \sigma/2\omega$, then:

$$F \equiv \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{(f^2-\sin^2\theta)}\frac{\partial}{\partial\theta}\right] + \frac{1}{(f^2-\sin^2\theta)}\left[\frac{i}{f}\frac{(f^2+\sin^2\theta)}{(f^2-\sin^2\theta)}\frac{\partial}{\partial\phi} + \frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{48}$$

When (46), (47), and (48) are substituted into (43), the resulting equation:

$$\chi_\sigma = \frac{\partial W_\sigma}{\partial z} + \frac{i\sigma}{4\omega^2 a^2} F\left(\frac{p_\sigma}{\rho_0}\right) \tag{49}$$

is identical to equation (23). The same procedure may now be followed, arriving at equations (34) and (38). However, the eigensolutions to (34) are now considerably more complex.

D. Eigensolutions for the Separation Coefficient h_n

In the first case, that of the planar model, the solutions to equation (34) may take on a very straightforward form. If the tidal variables have the periodic form:

$$\Psi_n = e^{i(\sigma T + k_\xi \xi + k_\eta \eta)} \tag{50}$$

then (34) becomes:

$$- \frac{a^2(k_\xi^2+k_\eta^2)}{(\sigma^2/4\omega^2 - 1)} \Psi_n + \frac{4\omega^2 a^2}{g h_n} \Psi_n = 0 \qquad (51)$$

or:

$$h_n = \frac{(\sigma^2 - 4\omega^2)}{g(k_\xi^2+k_\eta^2)} \qquad (52)$$

In the broadest sense of the planar model, that is in the case of a plane that is simply infinite, the wave numbers k_ξ and k_η, and hence h_n, can take on all real values. (h_n will be confined to positive or negative values, depending on whether $\sigma \gtrless 2\omega$). If the model is made more earthlike by assuming the geometry to be periodic in ξ and η, then the wave numbers and the eigenvalues of h_n must take on discrete values.

In the case of a rotating sphere, Ψ_n is no longer a simple function. It may be described by:

$$\Psi_n(\theta,\phi) = \Theta^s_{\sigma n}(\theta)\, e^{is\phi} \qquad (53)$$

The solutions for $\Theta^s_{\sigma n}$ obtained from (34) and (53) were first obtained by Hough, (1897); accordingly they are known as Hough's functions. The subscript σ denotes the oscillational frequency, s the number of wavelengths around a latitude circle, and n-s the number of nodes in the Hough's function, (and the pressure) between but not including the poles.

For each eigensolution of (34) there is a specific $\Theta^{s}_{\sigma n}$ and a corresponding value of h_n . Some typical values taken from Siebert, (1961), and Eckart, (1960) for the semidiurnal tide are given in Table I. σ is expressed as a multiple of ω. .

Table I. Semidiurnal Hough's Functions.

$\Theta^{s}_{\sigma n}$	h_n (km.)
Θ^{1}_{21}	15.6
Θ^{1}_{22}	5.70
Θ^{1}_{23}	2.94
Θ^{2}_{22}	7.85
Θ^{2}_{23}	3.77
Θ^{3}_{23}	4.85
Θ^{2}_{24}	2.11
Θ^{2}_{26}	0.96

For $\sigma < \omega$ and $s > 0$, that is, for slowly westward moving waves, a second class of Hough's functions exist. This second class has much larger values of h_n than its counterparts in the first class. Only one mode of the second class, Θ^{1}_{11} may be excited by the diurnal tide, and that only because the solar day is slightly shorter than the sidereal day. No oscillation of the second class may be excited at the semidiurnal frequency.

In both geometries, h_n decreases rapidly as the wavelength decreases in either zonal or meridional direction. A more complete discussion of Hough's functions and their application to the tides will be found in Kertz, (1957).

E. The Vertical Wave Equation

While equation (34) determines the horizontal character of the tides, equation (38) specifies its vertical character: whether it more nearly resembles internal or surface gravity waves, and whether there are resonances or vertical energy propagation. As it is a second order differential equation, two boundary conditions for y_n must be specified. As H_0 may vary in a complicated manner with height, the solution of (38) may be possible only by the use of simplified vertical temperature profiles, or by numerical methods.

The most drastic simplification is to assume an isothermal atmosphere, with constant H_0 , and to assume the heating function, J_σ , is everywhere zero. Two types of waves then exist, (c.f. Queney, 1947, Eckart, 1960, Hines, 1961). Let two new variables be defined as:

$$\mu_n^2 \equiv \frac{1}{4}\left[1 - \frac{4}{h_n}\left(\kappa H_0 - \frac{dH_0}{dz}\right)\right] \tag{54}$$

and:

$$\lambda_n^2 \equiv \frac{1}{4}\left[\frac{4}{h_n}\left(\kappa H_0 - \frac{dH_0}{dz}\right) - 1\right] \tag{55}$$

In an isothermal atmosphere, μ_n and λ_n are constant. If $\mu_n^2 > 0$, there will be solutions of the form:

$$y_n \sim e^{\pm \mu_n x} \tag{56}$$

While if $\lambda_n^2 > 0$, the solutions take the form:

$$y_n \sim e^{\pm i \lambda_n x} \tag{57}$$

Solutions of form (56) are surface gravity waves, and do not propagate energy vertically. Solutions of form (57) are internal gravity waves and transmit energy vertically as well as horizontally.

Now consider a two-layer atmosphere consisting of a warm isothermal lower layer and a cold isothermal upper layer. For a range of h_n, waves will be internal waves in the lower layer and surface waves in the upper layer. Energy propagated vertically in the lower region will be reflected, both at the earth's surface and at the interface. If the period needed for a wave to propagate from the ground to the interface and back is an integral harmonic of the period of a tidal driving force, strong resonance amplification can result. This, in considerable oversimplification, is the modern tidal resonance theory.

There is an approximation associated with Wentzel, (1926), Kramers, (1926), and Brilouin, (1926), which has been of considerable use to modern physics. This is commonly known as the WKB approximation. Eckart, (1960), has shown how this approximation can be applied to gravity waves under the appropriate circumstances. His development, applied to the problem at hand, is outlined below.

Without loss of generality, one may write:

$$y_n = Y_n(x)\, e^{i \Phi_n(x)} \tag{58}$$

where Y_n and Φ_n are real, non-linear functions of x.

Then:

$$\frac{dy_n}{dx} = i\, y_n \frac{d\Phi}{dx} + \frac{dY_n}{dx} e^{i\Phi} \tag{59}$$

The WKB approximation is made by assuming the second term on the right-hand side of (59) is negligible compared to the first, and that therefore:

$$\frac{dy_n}{dx} \simeq i\, y_n \frac{d\Phi}{dx} \tag{60}$$

If this is so, equation (38) reduces to:

$$-y_n\left(\frac{d\Phi}{dx}\right)^2 + \lambda_n^2 y_n = \frac{\kappa J_n}{\gamma g h_n} \tag{61}$$

If $J_n = 0$:

$$\begin{aligned} \Phi &= \int_{x_0}^{x} \lambda_n \, dx' \\ &= \int_{z_0}^{z} \frac{\lambda_n}{H_0} \, dz' \end{aligned} \tag{62}$$

The phase of y_n is determined to an arbitrary constant specified by x_0 or z_0 . For convenience x_0 or z_0 may be taken to be zero, with the phase incorporated into a complex Y_n .

In the atmosphere, this approximation becomes increasingly valid as wavelengths become shorter and changes in λ_n become small over a wavelength in the vertical; that is, the approximation is valid when:

$$\frac{d \ln \lambda_n}{d x} \ll 1 \tag{63}$$

It is not applicable to the surface gravity waves, nor in general to the primary tidal wave. The approximation has the effect of transforming the vertical coordinate into a new coordinate system in which the vertical wavelength is constant. In this new coordinate system, waves are propagated as they are in an isothermal atmosphere.

If there are abrupt changes in λ_n , a second approximation may also be used; this is to assume λ_n uniform except at a discontinuity surface at some height. This case has been treated by Queney, (1947), it is quite analogous to the reflection of electromagnetic waves at the interface between media of different index of refraction.

An internal gravity wave of the form:

$$y_n = A e^{i \lambda_n x} \tag{64}$$

has a downward phase propagation, but an upward energy propagation and group velocity. Let such a wave propagate energy up to the interface from below. Assume also a reflected wave:

$$y_n = B e^{-i \lambda_n x} \tag{65}$$

in the lower region, and a transmitted wave:

$$y_n = C e^{i \lambda_n' x} \tag{66}$$

in the upper region. Without loss in generality, the interface may be taken to be at $x = 0$. If discontinuities in pressure and vertical velocity are to be avoided, both y_n and $\frac{dy_n}{dx}$ must be continuous across the interface. Therefore:

$$A + C = B \tag{67}$$

$$A - C = \frac{\lambda_n'}{x_n} B \tag{68}$$

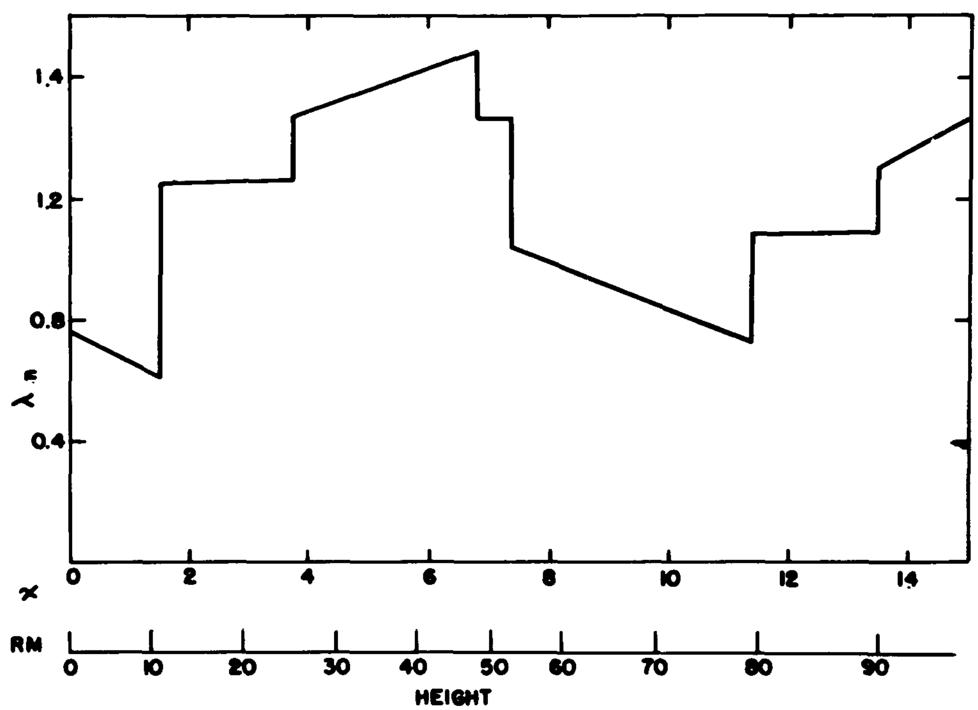

Figure 1. λ_n As a function of height, for h_n = 1 km., based on ARDC 1959 model atmosphere.

If $\frac{\lambda_n}{\lambda_n'} = \frac{1}{2}$, then $C = A/3$. As the energy flux is proportional to the square of the wave magnitude, only 11% of the energy is reflected.

Figure 1 shows λ_n as a function of χ for h_n = 1 km., based on the ARDC 1959 model atmosphere, (Minzner, Champion, and Pond, 1959). As this model has discrete jumps in the temperature gradient, and thus in the scale height gradient, it has discontinuities in λ_n . The largest of these appears at the tropopause, where $\frac{\lambda_n}{\lambda_n'} \simeq \frac{1}{2}$, and only 11% of the energy will be reflected in an upward propagating wave. (The ratio $\frac{\lambda_n}{\lambda_n'}$ is approximately constant for waves with length less than five or six thousand kilometers.) As the atmospheric changes are not discontinuous, this is a pessimistic estimate. Therefore, it seems internal gravity waves will be transmitted with little reflection. This conclusion does <u>not</u> take into account the effects of winds and wind shears, which Charney and Drazin, (1961) showed reflect Rossby waves quite strongly. This is an important possibility, which should be investigated in the future.

F. Transport of Tidal Energy

As the tidal motions are wave motions, they can transport energy from one part of the atmosphere to another. Consider a specific mass of air, in which the kinetic energy, potential energy, and internal energy per unit volume are given by $\frac{1}{2}|\vec{V}|^2\rho$, $\Omega\rho$, and $E\rho$, respectively. It can be shown that the rate of change of the total energy of this mass is, (c.f. Milne-Thomson, 1959):

$$\frac{D}{Dt}\int\left(\frac{1}{2}\rho|\vec{V}|^2+\rho\Omega+\rho E\right)d\tau=\int p\,\vec{V}\cdot\vec{n}\,dS \qquad (69)$$

where $d\tau$ is a differential volume, dS a differential surface element, and $\vec{n}$ a unit vector directed normally inward. Equation (69) states that the rate of change of the total energy of the mass is equal to the rate at which work is done on its boundry.

This equation may be integrated over a tidal period, to obtain the mean rate of change of energy, but it is necessary to use Lagrangian, rather than Eulerian parameters in evaluating the work done on the boundary. The excursions of the surface are quite small compared to the distances over which tidal velocities change, and the substitution of velocities observed at a point in space involves no significant error. The substitution of Eulerian for Lagrangian pressures needs further justification, however. Let $\wp$ be the pressure measured at a given particle. Then from equation (18),

$$i\sigma\mathcal{P}_\sigma = i\sigma P_\sigma + w_\sigma \frac{\partial P_0}{\partial z} \tag{70}$$

or:

$$\mathcal{P}_\sigma = P_\sigma - \frac{w_\sigma g \rho_0}{i\sigma} \tag{71}$$

First consider the mass of air which, at rest, would lie above a horizontal surface at some specified height. Under the influence of tides, the surface will oscillate about this position. The time average of the work done per unit area at any point on the surface is:

$$\frac{1}{2}\mathcal{R}e\left[w_\sigma^* \mathcal{P}_\sigma\right] \tag{72}$$

But since:

$$\mathcal{R}e\left[\frac{w_\sigma w_\sigma^*}{i}\right] = 0 \tag{73}$$

$$\frac{1}{2}\mathcal{R}e\left[w_\sigma^* \mathcal{P}_\sigma\right] = \frac{1}{2}\mathcal{R}e\left[w_\sigma^* P_\sigma\right] \tag{74}$$

While the vertical velocity causes a difference between P_σ and $\mathcal{P}_\sigma$, it is a difference whose effect is cancelled in averaging over a period.

Next consider the mass which, at rest, would lie to the north of a given latitude. With a tide, this surface will be displaced; however, an area on this surface will be practically equal to its meridional

projection. Since the displacement of the mass surface in the horizontal is small compared with the scale of horizontal tidal variations, the pressure and meridional velocity at a point on the mass surface will also be very nearly the same as at its horizontal projection on the latitude surface. Thus:

$$\frac{1}{2} \mathcal{R}e \left[V_\sigma^* \mathcal{P}_\sigma \right] \simeq \frac{1}{2} \mathcal{R}e \left[V_\sigma^* P_\sigma \right] \tag{75}$$

a similar argument can be made for zonal transport of energy across a meridional circle.

Lorenz, (1955), defines available potential energy as the potential and internal energy that is available for conversion to kinetic energy under any adiabatic redistribution of mass. Consider a given mass of fluid, constrained by pressure forces on its surface and by potential forces such as gravity. If the shape of the mass remains unchanged, and its center of gravity remains at the same potential. Then the resultant of the pressure forces acts to change the total momentum of the mass, and the integral of $p\vec{V}$ over the surface equals the rate of change of kinetic energy of the mass. Since the process is adiabatic, it is also the negative rate of change of available potential energy plus kinetic energy of the rest of the universe. As kinetic energy is not advected over the surface of the mass, $p\vec{V}$ may be taken to be a flux of available potential energy.

If the rigid mass moves to a different potential, not all of the pressure work will be converted to kinetic energy. However, the changes of potential energy may be recovered as kinetic energy at a later data,

and thus represents an increase in available potential energy. Similarly, if the body changes shape, there will be a change in internal energy. One may think of the pressure forces imparting kinetic energy to the surface layer of molecules, which in turn transfer the energy to the interior adiabatically, in part as internal and potential energy. Since the process is adiabatic and hence reversible, kinetic energy may be recovered. The stored energy is thus available potential energy, and $p\vec{V}$ may still be treated as a flux of available potential energy.

It has been assumed for this chapter that the only variations of atmospheric parameters in a horizontal plane are those of the tidal perturbations, therefore, the only available potential energy is that which may be associated with these perturbations. If one views tidal transports from an Eulerian point of view, advection of sensible heat and potential energy may occur. Such advections, as they may be related to the mean flux of tidal available potential energy, are discussed in Chapter VII.

CHAPTER III. BOUNDARY CONDITIONS FOR THE LINEAR TIDAL EQUATIONS

A. The Upper Boundary Condition

Equation (II-38) is a second order differential equation for $y_n(x)$; it requires two boundary conditions on y_n in order to obtain a complete solution. There has not been complete unanimity on the proper choice of these conditions. The lower boundary condition has normally been taken to be that $W_n = 0$ at the surface of the earth. This will be discussed in the second section of this chapter.

A very natural second choice is to require the total kinetic energy in a unit column of the atmosphere to be finite. That is:

$$\int_0^\infty \rho_0(x) |\vec{V}|^2 H_0(x)\, dx < \infty \tag{1}$$

It may be shown, (Siebert, 1951), that this is equivalent to requiring that:

$$\lim_{x \to \infty} \left[y_n(x) \cdot \sqrt{x} \right] = 0 \tag{2}$$

This boundary condition is adequate if waves behave like surface gravity waves, in which case it specifies the solution that attenuates with height. It is not applicable to isothermal atmospheres, or to atmospheres in which the upper regions are isothermal or allow the WKB approximation, when wavelengths are short enough for the waves to behave as internal gravity waves. Such waves do not attenuate with height. The fact that no such solutions

ground the sum of such transports must by matched by a downward flux in the primary tidal wave.

If it is assumed that the internal gravity waves generated by the tidal-terrain interaction have short wavelengths, and additional approximate relationship may be derived for isothermal atmospheres, or atmospheres where the WKB approximation is valid. Since these waves propagate energy upward:

$$\frac{d y_n}{d x} = i \lambda_n y_n \tag{35}$$

For short waves, $\lambda_n \propto k_n$, and $h_n \propto k_n^{-2}$. Therefore for large k_n :

$$W_n \simeq \gamma H_0 e^{x/2} y_n \tag{36}$$

$$P_n \simeq \frac{\gamma h_n \lambda_n}{\sigma} e^{-x/2} y_n \tag{37}$$

Thus P_n and W_n are approximately in phase for each of the internal gravity waves.

At any point on flat, oceanic portions of the surface of the earth, the sum of the vertical velocities of the tidal and secondary waves must be zero, and so must the total vertical energy flux. Since the pressures and vertical velocities of the secondary waves are in phase, the time of maximum vertical flux of these waves is in phase with their maximum vertical

velocity, and thus in phase with both the maximum downward flux and downward velocity of the tide. This requires that the tidal vertical velocity be 180 degrees out of phase with the tidal pressure.

With this relationship established, it is possible to examine qualitatively the effect of introducing terrain in tidal models. The basic tidal equation is:

$$\frac{d^2 y_n}{dx^2} - \frac{1}{4}\left[1 - \frac{4}{h_n}\left(\kappa H_0 + \frac{dH_0}{dx}\right)\right] y_n = \frac{\kappa J_n e^{-x/2}}{\gamma g h_n} \tag{38}$$

The complete solution consists of a particular solution to (38), plus general solutions to the corresponding homogeneous equation, of proper magnitude and phase to match the boundary conditions. One may divide the latter into a component that would be required to satisfy a flat lower boundry, and a remainder associated with the influence of terrain.

For waves such as the Θ_2^2 mode, w lags P by 90 degrees in the homogeneous solutions at ground level. Let P_a be the complete solution for the pressure at the ground with a flat surface, and let w_b and P_b be the vertical velocity and pressure of the additional homogeneous solution required to match an undulation surface. If w_b lags P_b by 90 degrees and $P_a + P_b$ by 180 degrees the phase relationship must be as shown in Figure 3.

The final ground pressure, $P_a + P_b$, must lag the pressure derived on the basis of flat ground. The actual atmosphere shows such a lag, of about 25 degrees, (Siebert, 1961).

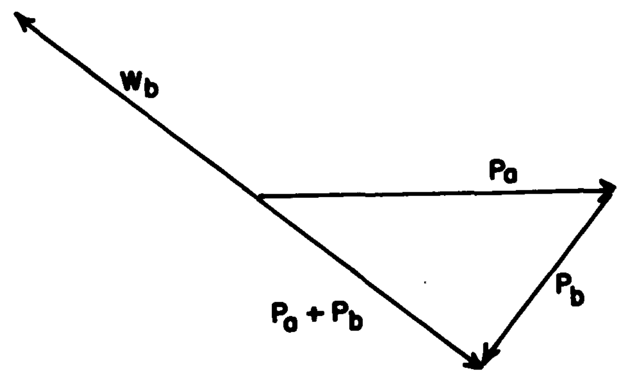

Figure 3. Phase relationships between pressure and vertical velocity at the ground.

C. Evaluation of the Vertical Velocities and the Total Energy Flux

Under certain circumstances, evaluation of the vertical velocities from equation (19) could be considerably simplified. All but the very largest secondary waves are internal gravity waves; if the ARDC 1959 Model Atmosphere temperature profile, (Minzner, Champion, and Pond, 1959) is used, then near the ground the ratio $|u_n| : |w_n|$ is about 100:1. It will be shown that typical values of $k_j A_j$ are a little more than 2×10^{-4}, so that the individual terms in the summation of (19) would be of the order

of 0.02 of the term w_n . If the number of such terms were limited, and u_ℓ and $A_{n-\ell}$ were known, the summation could be ignored, and each of the equations (19) could be solved separately for the corresponding w_n . If the number of terms for which $k_j A_j$ is substantial were great, however, a large collection of coupled equations would need to be solved.

A Fourier analysis of terrain height around latitude 35 degrees North was made in order to estimate the magnitudes of $k_j A_j$ out to wave number 240. Heights were obtained from the U.S. Army Map Service Series 1300 topographic maps and aeronautical maps, being taken at every quarter of a degree. The resulting amplitudes of $k_j A_j$ show no sign of decreasing out to wave number 240. (see Figure 4). Thus there are at least several hundred terms of potential significance in each summation, possibly many more.

If the phases of these terms were randomly oriented, the expected value of the summation would increase as the square root of the number of terms. Thus, 400 terms of order 0.02 w_n might be expected to sum to a value of 0.4 w_n . However, A_j and u_ℓ are not unform in magnitude, but have a substantial variance of their own; this would act to increase the magnitude of the sum, as would the unknown further extent of appreciable terrain height oscillation to higher wave numbers.

In the summation were to approach w_n in magnitude, the latter's size would be increased, on the average. But this would entail a further increase in w_n , in a progression which rapidly would become unstable.

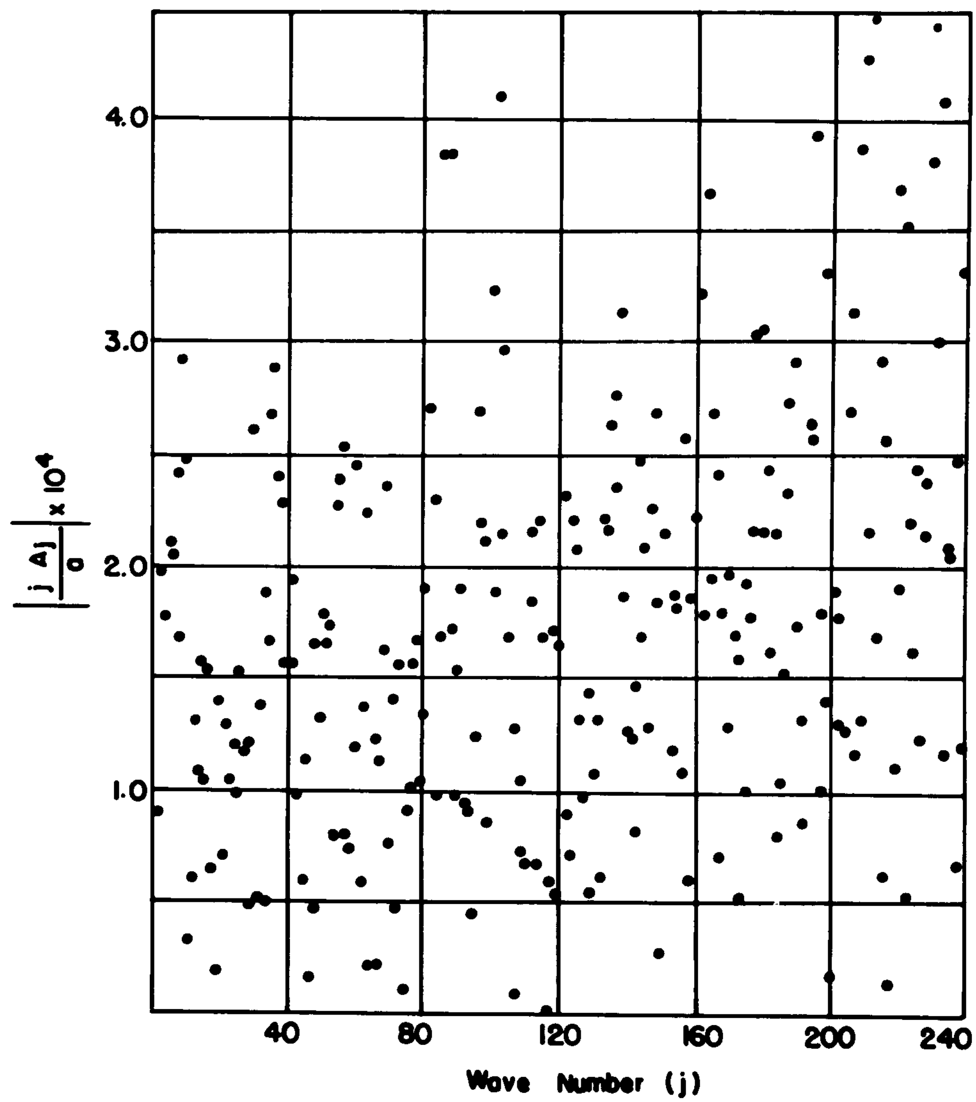

Figure 4. $\left|\frac{j A_j}{a}\right|$ as a function of wave number j, around latitude 35°N.

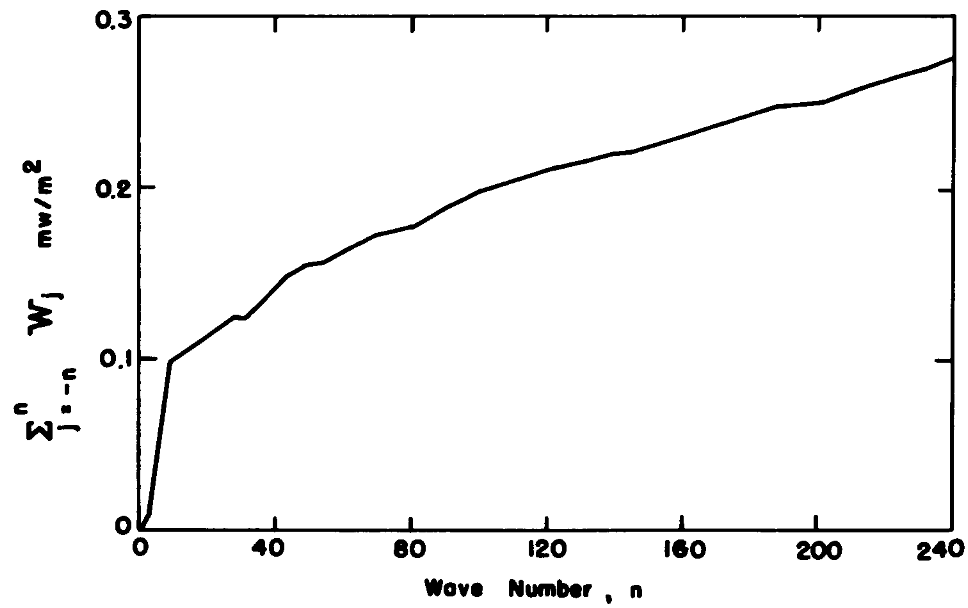

Figure 5. Summation of secondary wave energy flux as a function of wave number.

The breakdown of the assumption of random phase would eventually serve as a limiting control, at a point where the equations are thoroughly coupled. Thus it is quite possible that a value of W_n computed by ignoring the summation may underestimate the real values by a considerable margin, at least on the average. Nevertheless, this is the only practical way to obtain values of W_n, and the resulting energy transport.

This has been done, using a value u_* = 0.2 m/sec; the energy flux so computed is presented in Figure 5. The vertical tidal energy flux out to wave number 200 is about 0.25 mw/m^2. This figure might be doubled, to

include the effect of wave generated by V wind components. What is not known is the extent to which the energy flux has been underestimated for the reasons given above, and the upper limit to which the summation should be extended.

If one assumes that the summation in equation (19) is actually large compared to the term involving the primary tidal velocity, a second, much more intuitive approach is available. In this case, the primary tide loses its special identity, and may be treated with all the other waves. Defining:

$$C_l^n \equiv i\, A_{n-l}\, k_{n-l} \frac{u_l}{w_l} \tag{39}$$

and noting that u_l/w_l may be obtained from the solutions for each mode, with the upper boundry condition taken into account, equation (19) may be written:

$$w_n = \sum_{l=-\infty}^{\infty} C_l^n\, w_l \tag{40}$$

If l extends over a large range, there is intuitively no reason to expect $|w_n|^2$ to be systematically greater or less than its equivalent for neighboring wave numbers, especially if the "neighborhood" is small compared to distances in wave number space over which the mean or average of $|C_l^n|$ varies appreciably. (This excludes the small scale statistical variation in $|C_l^n|$). It would be logical to assume

that a sample of $|W_n|^2$ in a neighborhood would provide an approximation to the real values in that neighborhood.

Unfortunately, only a sample of one, the primary tidal oscillation, is available. Anticipating the results of the following chapter, this is:

$$|W_2|^2 \simeq 8 \times 10^{-8} \; m.^2/sec.^2 \tag{41}$$

If this is taken as a representative value for all waves of moderate wavelength, then from equation (34):

$$W_n \propto \frac{5 \times 10^{-3} \times 8 \times 10^{-8} \times a}{n} \simeq \frac{2 \times 10^{-3}}{n} \; w./m.^3 \tag{42}$$

If this is summed over the neighborhood to $\pm m$, omitting waves shorter than wave number four, (which are not internal gravity waves);

$$\sum_{\substack{n=-m \\ |n|>3}}^{m} \simeq 4 \times 10^{-3} \sum_{n=4}^{m} \frac{1}{n}$$

$$\simeq 4 \times 10^{-3} \ln\left(\frac{m}{4}\right) \quad w./m.^3 \tag{43}$$

If $m \simeq 200$, a flux of 15 mw/m^2 would be expected. Because of the logarithmic factor, the choice of a value for m is not overly critical. Again, one might double this figure to allow for the V wind component.

The crudity of this argument is self-evident. It is based on an intuitive argument and a statistical sample of one. It is presented not as a proof, but simply as an order of magnitude measurement, whose chief value may lie in stimulating others to more rigorous analyses. There is some self-consistency, since a typical value of $A_{n-g} R_{n-g} U_g$ is about 0.4 x 10^{-4} m/sec, compared to values of W_n here assumed to be $\sim$ 3 x 10^{-4} m/sec.

CHAPTER IV. VERTICAL ENERGY TRANSPORT IN THE OBSERVED TIDE

A. The Vertical Energy Flux - Terciera

Harris, Finger, and Teweles, (1962) have analyzed radiosonde observations at Terciera, in the Azores, obtaining the tidal components of wind, pressure, and temperature. Under certain assumptions, it is possible to analyze this data to compute the vertical flux of tidal energy, and to compare it with the losses predicted for the tidal-terrain interaction in Chapter III. The primary assumption is that the data represents only one mode of oscillation, with a known separation factor, h_n.

Ground level pressure observations show that one mode, the $\Theta^{2}_{2,2}$ mode is an order of magnitude greater than other modes, (Siebert, 1961) at least in moderate and low latitudes. (In high latitudes, the $\Theta^{0}_{2,3}$ mode becomes important.) For this mode h_n = 7.85 km. In the stratosphere, insolational heating is small, and for this mode:

$$\frac{1}{4}\left[\frac{4}{h_n}\left(\kappa H_0 + \frac{dH_0}{dx}\right) - 1\right] \simeq 0 \qquad (1)$$

so that to a first approximation, equation (II-38) becomes:

$$\frac{d^2 y_n}{d x^2} \simeq 0 \qquad (2)$$

In an isothermal region, $\left|\frac{dy_n}{dx}\right| \propto |y_n|$ and therefore:

$$P_n e^{x/2} \propto y_n \qquad (3)$$

from equation (II-39). Table I shows the observed amplitude and phase of $P_\sigma e^{x/2}$ as a function of height in the stratosphere.

Table II. $P_\sigma e^{x/2}$ for Terciera.

Height, mb.	Amplitude, n/m^2	Phase, degrees
200	31.3	25
175	31.0	21
150	28.9	30
125	31.0	18
100	28.5	16
80	31.8	20
60	28.6	18
50	31.3	26
40	30.0	22
30	28.8	25
20	28.4	16

The constancy of $P_\sigma e^{x/2}$ is in good accord with equation (2), if it is assumed that the $W_{2\,2}^{2}$ mode also dominates the stratosphere. No other single mode could produce the uniformity observed in Table I. While it is possible that several modes combine to produce the observed

pressure, it would be a singularly unfortunate combination that could produce such a result. It will therefore be assumed that the mode dominates throughout the range of heights in the Terciera data.

In this case, the observed pressure may be treated as:

$$P_r = \frac{\gamma h_n P_o(0)\, e^{-x/2}}{i \sigma H_o(x)} \left[\frac{dy_n}{dx} - \frac{1}{2} y_n \right] \Theta_{2}^{2} \qquad (4)$$

For convenience, Θ_{2}^{2} may be normalized to a value of unity at the observation site, incorporating a multiplicative factor into y_n . Therefore:

$$\left[\frac{dy_n}{dx} - \frac{1}{2} y_n \right] = \frac{i P_r(x) H_o(x)\, e^{x/2}}{\gamma h_n P_o(0)} \qquad (5)$$

is an experimentally observed quantity, and y_n may be determined if a function of y_n is known at any height.

The traditional approach is to take $W_n(0) = 0$, so that:

$$\left[\frac{dy_n}{dx} + \left(\frac{H}{h_n} - \frac{1}{2}\right) y_n \right]_{x=0} = 0 \qquad (6)$$

However, the tidal-terrain effect precludes this. Were the nature of the effect well enough understood to relate P_n to W_n , a boundary condition might be obtained. While the relative phase of these parameters was estimated, their relative magnitude was not.

One alternative approach is through equation (III-2):

$$\lim_{x \to \infty} [y_n(x) \cdot \sqrt{x}] = 0 \tag{7}$$

which is derived from the condition that the kinetic energy in a unit column be finite. In the present case, this equation is more useful in ruling out solutions, rather than establishing an exact solution. Examination of the Terciera data, and very high level data at 80 to 100 km, (Greenhow and Neufeld, 1955), show that (5) is a function whose amplitude decreases continuously with height. If at any height, $|\frac{1}{2}y_n| \gg |\frac{dy_n}{dx} - \frac{1}{2}y_n|$ then

$$\frac{dy_n}{dx} \simeq \frac{1}{2}y_n \tag{8}$$

or:

$$y_n \propto e^{x/2} \tag{9}$$

from that height upward. Equation (7) then confines $|\frac{1}{2}y_n|$ to values of the order of $|\frac{dy_n}{dx} - \frac{1}{2}y_n|$. If condition (6) is applied to the Terciera data, y_n is exploding exponentially at the tropopause and above.

A second possibility, in line with the comments that have been made in this chapter, is to assume y_n to be constant in the stratosphere, say at 100 mb. With the aid of this assumption, y_n has been computed graphically, and the vertical velocity, W_n computed from equation (II-40).

(The pressure and velocity are presented in Table III.)

Table III. Pressure and vertical velocity for the semidiurnal tide - Terciera.

Height (x)	Height (mb)	Pressure n/m^2	Pressure degrees	Vertical Velocity m/sec	Vertical Velocity degrees
0.0	1000	50	65	2.89×10^{-4}	254
0.2	830	46	59	3.19	255
0.4	690	38	55	3.41	258
0.6	555	33	44	3.48	260
0.8	450	27	40	3.50	265
1.0	365	21	37	3.55	270
1.2	305	18	29	3.62	274
1.4	250	16	28	3.65	277
1.6	205	14	25	3.90	280
1.8	170	13	21	4.33	283
2.0	135	11	20	4.88	287

The ARDC 1959 Model atmosphere was used in computing χ . Figure 6 shows the vertical flux of tidal energy, $\frac{1}{2} \mathcal{R}e \left[w_n^* P_n \right]$ computed from these values.

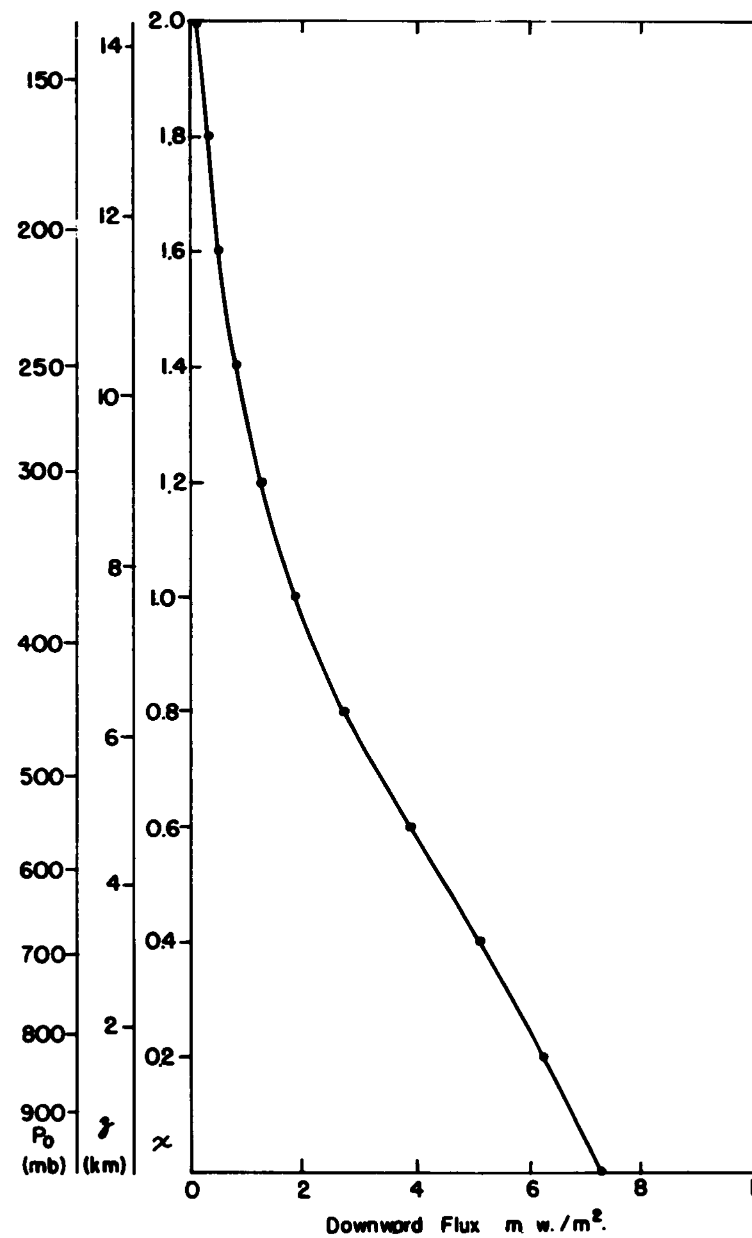

Figure 6. Downward flux of energy in the semidiurnal tide - Terciera.

There is a downward flux of tidal energy that reaches a maximum of 7×10^{-3} w/m^2 at the ground. The divergence of this flux requires a tropospheric source of energy that will be discussed in the following section. In Chapter III it was established that the pressure and vertical velocity of the primary tide should be approximately 180 degrees out of phase at the ground. The observed phase difference is 189 degrees.

It is worth noting that the total pressure variation of the secondary waves should be small compared to the variation of the primary wave. From equation (III-22), the vertical energy transport of a secondary wave is:

$$W_n = \frac{H_0(x)\,\sigma\,|P_n(x)|^2}{2\,P_0(x)}\,\mathrm{Re}\left[\frac{i\lambda_n + H_0/h_n - 1/2}{\lambda_n + i/2}\right]$$

$$\simeq \frac{H_0^2(x)\,\sigma\,|P_n(x)|^2}{2\,P_0(x)\,\lambda_n\,h_n} \tag{10}$$

Since h_n varies as the square, and λ_n inversely as the first power of horizontal wavelength, short waves will require a smaller value of to transport a given energy flux. Therefore:

$$\sum_n |P_n(0)|^2 \tag{11}$$

summed over the secondary waves will be smaller than the square of the magnitude of the primary pressure wave, much smaller if the bulk of the

secondary energy is transported by waves of a few hundred kilometers length.

But if the phases of the secondary waves are randomly oriented, which would seem a logical assumption for a large number of waves, the square of the total secondary pressure variation will equal the sum of the squares of the individual waves. Thus the total secondary pressure variation should be small compared to the primary variation, and would not be likely to be observed in the data.

B. Generation of Available Potential Energy by the Tide - Terciera

The preceding section has shown that:

$$\frac{\partial}{\partial z} \mathcal{R}e\left[W_n^* P_n\right] \equiv \frac{\partial}{\partial z} \mathcal{R}e\left[W_\sigma^* P_\sigma\right] > 0 \tag{12}$$

in the troposphere. Unless there is a similar convergence of the flux ($V_\sigma^* P_\sigma$) in the horizontal, (and Chapter VII will show this is probably small), there must be a mean generation of available potential energy. Lorenz, (1955), has shown that available potential energy is generated by conversion from kinetic energy, or by differential heating. Since the mean value of tidal kinetic energy of a given mass does not change, the latter process must be operative. The mean rate of available potential energy generation by heating per unit mass is:

$$\frac{\Gamma_d \ \mathcal{R}e\left[T_\sigma' J_\sigma^{*\prime}\right]}{2 T_0 \left(\Gamma_d - \Gamma_0\right)} \tag{13}$$

where:

$$\Gamma_d \equiv g/c_p$$

$$\Gamma_0 \equiv -\frac{\partial T_0}{\partial z} \tag{14}$$

are the adiabatic and actual lapse rates. Primes refer to quantities measured at a given particle, Lagrangian, rather than Eulerian parameters.

From the first law of thermodynamics, (see II-17):

$$\frac{\gamma g H_0}{\rho_0}\frac{D\rho_\sigma}{Dt} = \frac{1}{\rho_0}\frac{DP_\sigma}{Dt} - (\gamma-1) J_\sigma' \tag{15}$$

with the aid of the perfect gas law, this may be reduced to:

$$\frac{P_0}{R\rho_0 T_0}\frac{DT_\sigma}{Dt} = \frac{1}{\rho_0}\frac{D\rho_\sigma}{Dt} + J_\sigma' \tag{16}$$

or, since Lagrangian tidal parameters will also have a time dependence $e^{i\sigma t}$;

$$\frac{P_0}{R\rho_0 T_0} T_\sigma' = \frac{\rho_\sigma'}{\rho_0} + \frac{J_\sigma'}{i\sigma} \tag{17}$$

If equation (17) is multiplied by $J_\sigma'^{*}$ and averaged over a cycle:

$$\frac{1}{2} \mathcal{R}e\left[J_\sigma^{*\prime} T_\sigma\right] = \frac{R T_0}{2 P_0} \mathcal{R}e\left[P_\sigma' J_\sigma^{*\prime}\right] \tag{18}$$

With the approximations used in developing the linear tidal theory:

$$P_\sigma{}' = \frac{1}{i\sigma}\frac{DP_\sigma}{Dt} = P_\sigma + \frac{W_\sigma}{i\sigma}\frac{\partial P_o}{\partial y} \tag{19}$$

$P_\sigma{}'$ may be computed from (19) and observed data, and if $J_\sigma{}'$ is known, the rate of generation of available potential energy may be computed. As J_σ does not vary rapidly with position, it is essentially the same as $J_\sigma{}'$, and may be substituted for it.

Siebert, (1955, 1961), used the empirical formula of Mügge and Möller, (1932), for water vapor absorption to evaluate insolational heating in the atmosphere as a source of the tides. Using a rather artificial model with a resonance amplification of 3.7, he obtained tides one-third the amplitude of the observed tides.

The Mügge-Möller equation was based on early experimental data by Fowle, (1915). While Fowle did excellent work, neither theory nor equipmer were very good at the time of his investigation. For example, he did not take into account the variation of absorptivity of water vapor and carbon dioxide with temperature and pressure. A more recent and complete study by Howard, Burch, and Williams, (1955, 1956), has been used by Roach, (1961), to compute mean daily heating rates in the atmosphere for the months of January, April, July, and October.

It is possible to derive the semi-diurnal heating component from Roach's computations. His data fit the Mügge-Möller approximation that

absorption at any given height varies as $\cos.^{0.7}\zeta$, where ζ is the solar zenith angle. (Numerical values, of course, differ.) The heating rate may then be expanded in a Fourier time series:

$$\frac{C_0}{2} + \sum_{\sigma} C_\sigma \cos \sigma x = \begin{cases} \cos^{0.7}\zeta & \text{IN THE DAY} \\ 0 \text{ AT NIGHT.} & \end{cases} \tag{20}$$

where, (Siebert, 1955, 1961):

$$C_\sigma = \frac{1}{\pi}\int_{-x_0}^{x_0} (\sin\theta \sin\delta + \cos\theta \cos\delta \cos x)^{0.7} \cos \sigma x \, dx \tag{21}$$

Here x_0 is the local time of sunrise or sunset and δ is the solar declination. The mean and semidiurnal coefficients of this series were computed for latitude 38 degrees North, and the four months used. The ratio of these coefficients was then used to obtain the semidiurnal heating coefficient at varying heights from Roach's mean data. These are shown for January, April, July, and October in Figure 7. Figure 8 shows the mean of these values, taken as the annual mean semidiurnal heating function.

With this, the values of equation 13 were computed for Terciera as shown in Figure 9. The values of:

$$\frac{\partial}{\partial z}\left[\frac{1}{2}\, \mathrm{Re}\,(w_\sigma^{*} p_\sigma)\right] \tag{22}$$

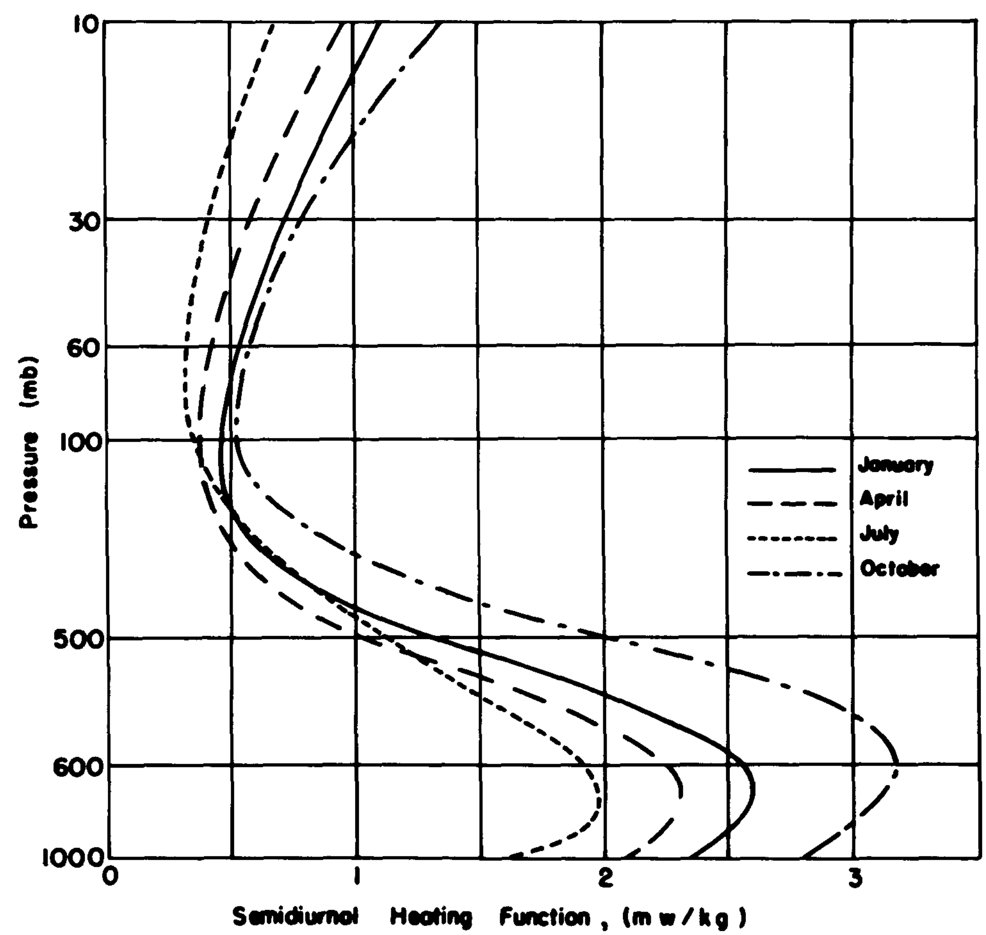

Figure 7. Semidiurnal heating function as a function of height for four months at latitude 38°N.

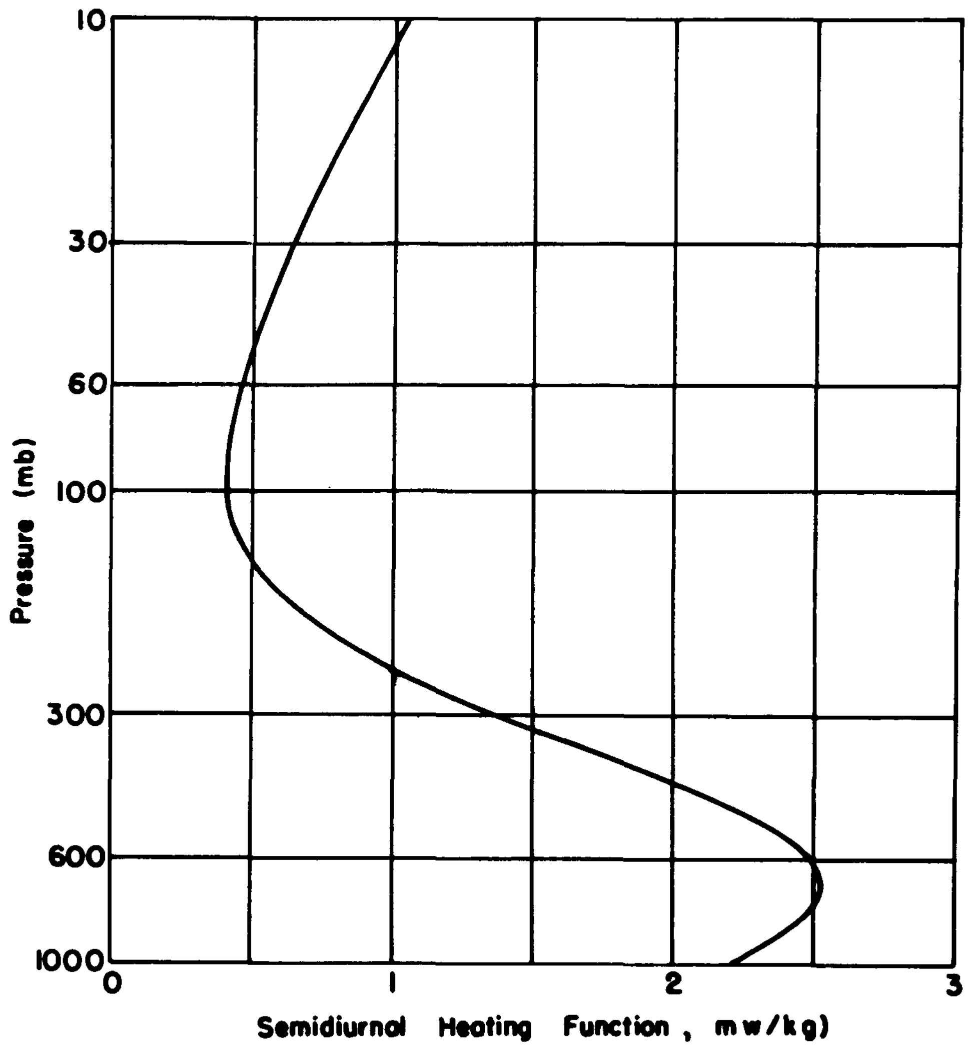

Figure 8. Annual mean semidiurnal heating function for 38°N, as a function of height.

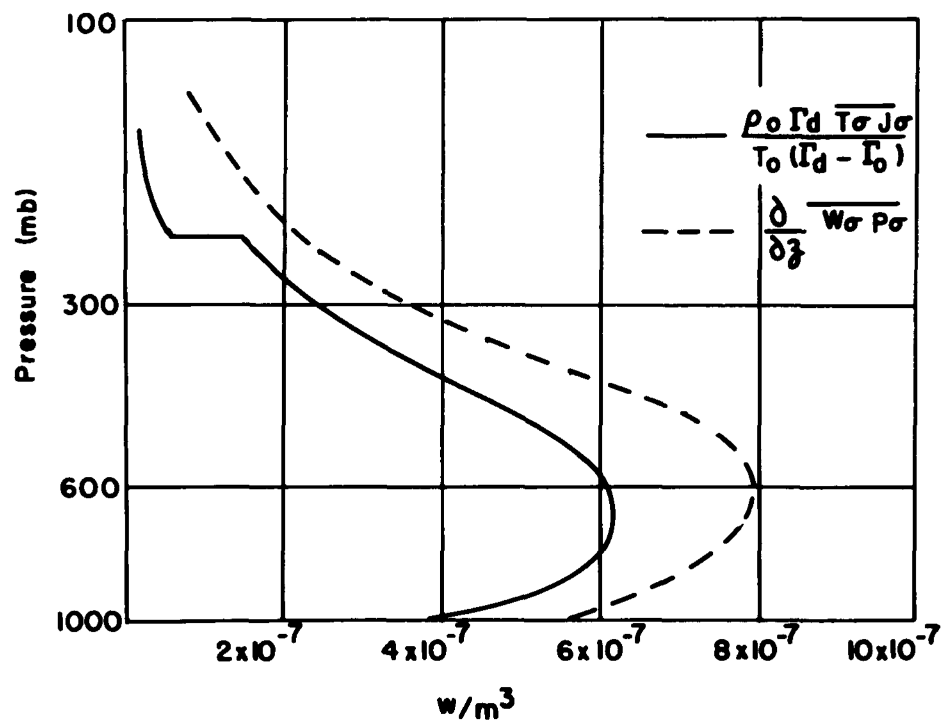

Figure 9. Divergence of vertical energy flux and rate of generation of available potential energy in the semidiurnal tide - Terciera.

are also shown. In view of the limitations of the data and the approximations that have gone into these computations, and in view of the possibility of horizontal divergence of the wave energy flux, the agreement must be considered to be satisfactory.

C. Comparison of Tidal Pressure Fluctuation - Terciera and Fort Worth

Messrs. Harris and Finger and Dr. Teweles, of the U. S. Weather Bureau, have very kindly made available a second set of tidal data, for Fort Worth, Texas, in advance of publication. A comparison of the pressures at Fort Worth and Terciera offers some measure of the degree to which the $\Theta_{2,2}^{2}$ mode is dominant. Because of the difference in latitudes, the Fort Worth pressure in this mode would be 1.42 times that of the Terciera data; the phases should be the same, when referred to local time.

Figure 10 shows the Fort Worth semidiurnal tide, divided by 1.42, and for comparison, the Terciera counterpart. (The 1000 mb level is below ground at Fort Worth.) Up to the 100 mb level, the agreement is quite good, especially with regard to amplitude. The Terciera data shows a phase angle perhaps 10 degrees larger at lower levels; this difference largely disappears at higher levels.

Above 100 mb, the Fort Worth data shows a comparatively rapid drop in tidal pressure, to a minimum at 50 mb. Pressure minima, or nodes, are quite possible in tidal theory; in fact, Pekaris predicted a node at about the 10 mb height. However, there are several features of the Fort Worth data that are not in harmony with such a node.

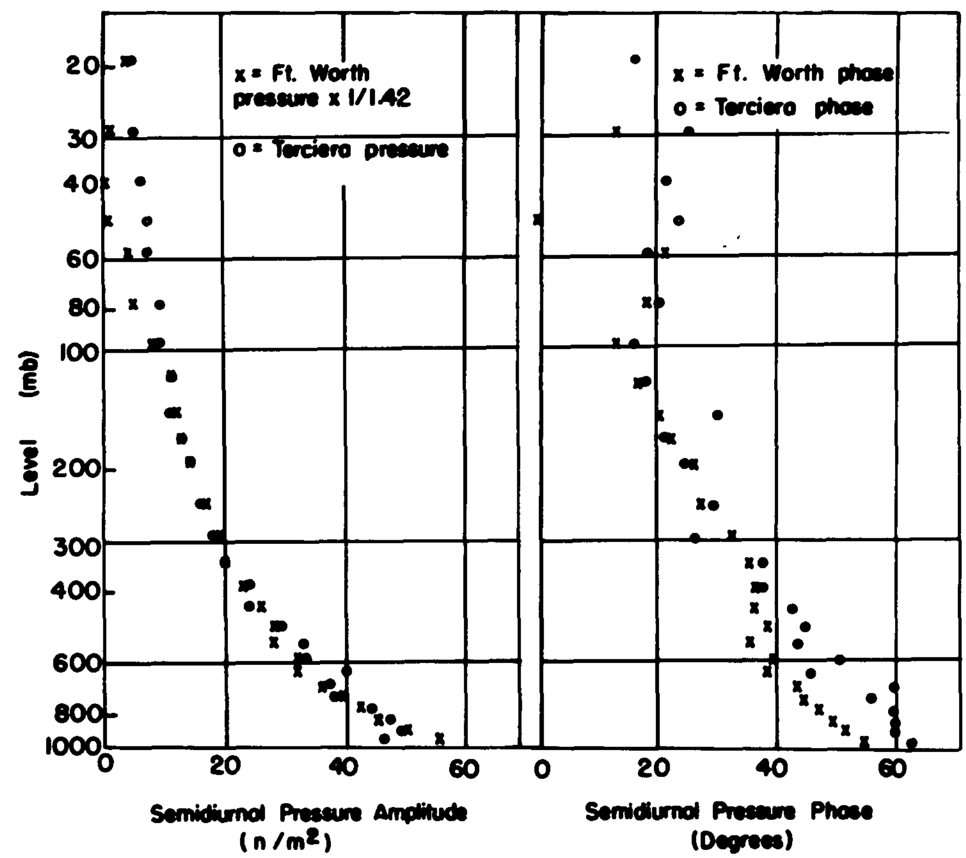

Figure 10. Comparison of the semidiurnal pressure fluctuations at Terciera and Fort Worth.

If a single mode of oscillation were dominant, as has been postulated, the phase of the pressure should reverse with height as a node is passed. The data shows no such phase reversal. Similarly, there should be a node and phase reversal of the tidal winds; observationally, the 50 mb northward wind drops to about half of its normal value, but all other winds show no sign of a node, either by amplitude or phase. It is, of course, possible that a combination of modes might lead to a pressure minimum. It is difficult to imagine such a combination that would not produce considerable fluctuation in amplitude and phases at lower levels, especially considering the rather abrupt onset of the 50 mb minimum.

Disregarding modal theory, the change in pressure fluctuation between the 60 and 50 mb levels implies a semidiurnal fluctuation of 5.5 meter amplitude in the thickness of this layer, normally 850 meters thick. This fluctuation of 0.65% in the thickness implies a similar fluctuation in the mean temperature of the layer, about 1.4 degrees. The observed semidiurnal temperature fluctuation at this height is only about 0.1 degrees, however, and the pressure and temperature data are not consistent. From the above, it seems more likely the pressure is in error.

CHAPTER V. VISCOSITY AND DAMPING

A. Viscosity in the Atmosphere

As it has been developed to date, tidal theory had been based on the assumption of an inviscid atmosphere. It is not difficult to develop an approximate theory to include effects of viscosity; though its value is limited by lack of knowledge of atmospheric viscosity, especially eddy viscosity. Nevertheless, the results are of importance to both primary and secondary tidal waves.

In the troposphere, eddy viscosity is far more important than molecular viscosity; it is also ill-defined, quite variable, and difficult to measure. Even when measurements are made, they may involve motions of different scales, and thus may be applicable to different problems.

Hess, (1959) quotes a values of $\nu = 5\ m^2/sec$, valid for the lowest kilometer. This is based on the rotation of the wind vector with height, as predicted by the Eckmann Spiral theory. Palmen, (1955) has analysed momentum transport of the zonal wind and has computed a value of 22 m^2/sec at 700 mb. Data on eddy diffusivity to be published by Prof. Newell, (1963) show mean values of eddy diffusivity of 3-30 m^2/sec in the troposphere, and values one or two orders of magnitude less in stratospheric regions. The eddy viscosity should be comparable.

On the basis of these data, a value of $\nu = 10\ m^2/sec$ will be assumed for the eddy viscosity, with the recognition that it may readily be in error

by a factor of two, possibly even a factor of ten. Fortunately, the results of the following equations depend on the square root and cube root of viscosity, so that the final errors are considerably reduced.

B. Mathematical Development - Planar Geometry

The model atmosphere used in the following discussion is planar, isothermal, and rotating; the equations are linearized, but a uniform viscosity in introduced. It is also assumed that the waves studied resemble secondary internal gravity waves in that they propagate energy upward, and have vertical shear far greater than horizontal shear, so that only the former will be included.

With these approximations the equations of horizontal motion become:

$$i \sigma u_n - 2\omega v_n = -\frac{1}{\rho_0}\frac{\partial P_n}{\partial \xi} + \nu \frac{\partial^2 u_n}{\partial z^2} \tag{1}$$

$$i \sigma v_n + 2\omega u_n = -\frac{1}{\rho_0}\frac{\partial P_n'}{\partial \eta} + \nu \frac{\partial^2 v_n}{\partial z^2} \tag{2}$$

In the inviscid solution for this geometry, the wave parameter y_n is proportional to $e^{i\lambda_n z/H_0}$ or to $e^{-\mu_n z/H_0}$, depending on whether the wave is of the internal or surface gravity wave type. It will assumed that in the present case, y_n has the form $e^{\Phi z/H_0}$, where Φ may be real, imaginary, or complex.

As u_n and v_n are linear functions of y_n and its derivative with respect to z, multiplied by $e^{z/2H_0}$, they are proportional to:

$$e^{(\frac{1}{2}+\Phi)z/H_0} \tag{3}$$

One further assumption, that:

$$|\Phi| \gg \frac{1}{2} \tag{4}$$

serves to simplify the equations; u_n and v_n may then be taken as proportional to:

$$e^{\Phi z/H} \tag{5}$$

If a new variable:

$$\sigma' \equiv \sigma + \frac{i\nu\Phi^2}{H_0^2} \tag{6}$$

is introduced, equations (1) and (2) become:

$$i\sigma' u_n - 2\omega v_n = -\frac{1}{\rho_0}\frac{\partial p_n}{\partial \xi} \tag{7}$$

$$i\sigma' v_n + 2\omega u_n = -\frac{1}{\rho_0}\frac{\partial p_n}{\partial \eta} \tag{8}$$

The solutions to the wave problem may then proceed formally as in the linearized inviscid solution, except that some variables that were real are now complex. The quantity ($\sigma'^2 - 4\omega^2$) is substituted for ($\sigma^2 - 4\omega^2$) and the separation parameter,   is therefore complex, since:

$$h_n = \frac{(\sigma^2 - 4\omega^2) + 2i\sigma\nu\Phi^2/H_0^2 - \nu^2\Phi^4/H_0^4}{k_\xi^2\, g} \tag{9}$$

The horizontal variation of the wave has been taken to be of the form:

$$e^{i k_\xi \xi} \tag{10}$$

In an isothermal atmosphere, the vertical wave equation is:

$$\frac{d^2 y_n}{dx^2} + \left[\frac{\kappa H_0}{h_n} - \frac{1}{4}\right] y_n = 0 \tag{11}$$

where:

$$x = \int_0^z \frac{dz'}{H_0} = \frac{z}{H_0} \tag{12}$$

This yields the relation:

$$\begin{aligned} \Phi^2 &= \frac{1}{4} - \frac{\kappa H_0}{h_n} \\ &= \frac{1}{4} - \frac{\kappa H_0 k_\xi^2 g}{(\sigma^2 - 4\omega^2) - \nu^2\Phi^4/H_0^4 + 2i\sigma\nu\Phi^2/H_0^2} \end{aligned} \tag{13}$$

or:

$$\frac{\nu^2 \Phi^6}{H_0^4} - \left(\frac{2 i \sigma \nu}{H_0^2} + \frac{\nu^2}{4 H_0^2}\right)\Phi^4 - \left[(\sigma^2 - 4\omega^2) - \frac{i \sigma \nu}{2 H_0^2}\right]\Phi^2$$

$$+ \frac{(\sigma^2 - 4\omega^2)}{4} - \kappa H_0 k_\xi^2 g = 0 \tag{14}$$

The following are taken as representative values which would make the model used here the closest counterpart of the real atmosphere for waves of semidiurnal frequency:

$$\begin{array}{ll} \sigma = 1.5 \times 10^{-4} \ \sec^{-1} & \nu = 10 \ m^2 \sec^{-1} \\ 2\omega = 1.0 \times 10^{-4} \ \sec^{-1} & g = 10 \ m \ \sec^{-1} \\ H_0 = 8 \times 10^{-3} \ m & \kappa = 2/7 \end{array} \tag{15}$$

Then:

$$\begin{array}{ll} \dfrac{2\sigma\nu}{H_0^2} = 4.7 \times 10^{-11} & (\sigma^2 - 4\omega^2) = 1.25 \times 10^{-8} \\ \dfrac{\nu}{4 H_0^2} = 6.1 \times 10^{-15} & \dfrac{\sigma\nu}{2 H_0^2} = 1.2 \times 10^{-11} \end{array} \tag{16}$$

If wavelengths are limited to those less than 3000 km, $k_\xi > 2 \times 10^{-6}$ and:

$$\kappa H_0 k_\xi^2 g > 9 \times 10^{-8} \tag{17}$$

while:

$$\frac{(\sigma^2 - 4\omega^2)}{4} = 0.31 \times 10^{-8} \tag{18}$$

Therefore, to a good approximation, equation (14) may be written:

$$\frac{\nu\Phi^6}{H_0^4} - \frac{2i\sigma\nu\Phi^4}{H_0^2} - (\sigma^2-4\omega^2)\,\Phi^2 - k H_0 R_\xi^2 g = 0 \tag{19}$$

This is a sixth order equation, which may readily be reduced to a cubic by the substitution:

$$S = 10^{-3}\Phi^2 \tag{20}$$

When (20) and the preceding numerical values are substituted into (19), the resulting equation is:

$$S^3 - i\,1.92\,S^2 - 0.512\,S - 0.902\times10^9 R_\xi^2 = 0 \tag{21}$$

For sufficiently small values of R_ξ^2, one root of this equation approaches:

$$S \simeq \frac{-0.902\times10^9 R_\xi^2}{0.512} \tag{22}$$

or in algebraic terms:

$$\Phi^2 \simeq \frac{-k H_0 R_\xi^2 g}{(\sigma^2-4\omega^2)} \tag{23}$$

This is simply the inviscid solution. The other roots are approximately:

$$S \simeq -i\,\frac{1.92}{2} \pm \sqrt{-\frac{3.67}{4} + .512} \tag{24}$$

$$= -i\,1.60\,,\ -i\,0.323$$

The modes of oscillation described by (24) correspond to viscously coupled waves of comparatively short wavelength and strong damping in the vertical. In the case of no rotation, they reduce to one mode, with:

$$\frac{\Phi^2}{H_0^2} = \frac{-2i\sigma}{\nu} \tag{25}$$

This is simply the classical solution to an oscillating plate in a viscous fluid. If σ, rather than ω goes to zero, the solution is an Eckman spiral.

Chapter III was based on an inviscid theory, and ignores the modes specified by (24). There are several reasons for doubting that viscous-coupled modes are excited to a large extent. First, eddy viscosity decreases near the ground; qualitatively this might be expected to decouple tidal-terrain effects from viscous waves. This is observed in the lower boundary layer beneath Ekman spirals, (Hess, 1959). Second, eddy viscosity varies greatly from one place to another and over times comparable to tidal times. Since the vertical wavelength of viscous waves is dependent on eddy viscosity, it will be difficult to maintain them on anything approaching a steady-state basis. Thirdly, the viscous theory presumes all tidal velocities, not simply the vertical go to zero at the ground, with a substantial variation of velocity over a scale of one km in the vertical. This is not observed in the Terciera data.

A somewhat better approximation to the gravity wave solution may be obtained by setting:

$$S = S_r + i S_i \tag{26}$$

where S_r and S_i are real. For small R_ξ^2 , $S_i \ll S_r$, and $S_r \ll 1$. If (26) is substituted into (21) and real and imaginary parts are separately compared:

$$S_r^3 - 3 S_r S_i^2 - 3.84 S_r S_i - 0.512 S_r - 0.902\times10^{9} R_\xi = 0 \tag{27}$$

$$3 S_r^2 S_i - S_i^3 + 1.92 S_r^2 - 1.92 S_i^2 - 0.512 S_i = 0 \tag{28}$$

If only the largest of the terms in (27) and (28) are retained:

$$S_r \simeq \frac{-0.902\times10^{9} R_\xi^2}{0.512} \tag{29}$$

$$S_i \simeq \frac{1.92}{0.512} S_r^2 \tag{30}$$

Algebraically, this is equivalent to:

$$\Phi^2 \simeq \frac{-k H_0 R_\xi^2 g}{(\sigma^2 - 4\omega^2)}\left[1 + \frac{i 2 \sigma \nu k g R_\xi^2}{H_0 (\sigma^2 - 4\omega^2)^2}\right] \tag{31}$$

or:

$$\frac{\Phi}{H} \simeq \frac{i k^{1/2} g^{1/2} R_\xi}{H_0^{1/2} (\sigma^2 - 4\omega^2)^{1/2}}\left[1 + \frac{i \sigma \nu k g R_\xi^2}{H (\sigma^2 - 4\omega^2)}\right] \tag{32}$$

Let:

$$k_z \equiv \frac{A^{1/2} g^{1/2} k_\xi}{H_0^{1/2} (\sigma^2 - 4\omega^2)^{1/2}} \tag{33}$$

Then:

$$\frac{\Phi}{H_0} \simeq i k_z \left[1 + \frac{i \sigma \nu k_z^2}{(\sigma^2 - 4\omega^2)} \right] \tag{34}$$

and:

$$y_n \propto e^{i k_z z} e^{- \frac{\sigma \nu k_z^2 z}{(\sigma^2 - 4\omega^2)}} \tag{35}$$

The mean upward flux of wave energy is proportional to $|y_n|^2$, or:

$$W_n \propto e^{- \frac{2\sigma \nu k_\xi^2 z}{(\sigma^2 - 4\omega^2)}} \tag{36}$$

If the energy flux is reduced to e^{-1} of that at the ground level value at 15 km, then with the assumed values of σ and ν :

$$k_z = 0.65 \times 10^{-3} \ m^{-1} \tag{37}$$

The approximations used in deriving these results break down when $\nu k_z^2 \rightarrow \sigma$, that is, when the viscous terms in the equations of motion are comparable to the other terms. This occurs when:

$$k_z \rightarrow 4 \times 10^{-3} \ m^{-1} \tag{38}$$

Within the range defined by (37) and (38), the waves should behave much as inviscid gravity waves, except for the vertical damping factor in (35). For shorter wavelengths, a complete viscous solution should be obtained.

There is another, more physical method of deriving the damping. This has been done in part by Hines, (1961), for the non-rotation planar case, again subject to the assumption that the waves are not seriously distorted by the viscosity. If horizontal shears are neglected, the rate at which kinetic energy is dissipated is:

$$R \simeq \nu k_z^2 u_n \cdot u_n^* \tag{39}$$

The total energy density associated with such waves in a non-rotating system is:

$$E \simeq \frac{1}{2} u_n \cdot u_n^* \tag{40}$$

One may associate with these a damping time:

$$\tau_D = \frac{E}{R} = \frac{1}{2 \nu k_z^2} \tag{41}$$

and a damping distance:

$$L_D = \frac{\tau_D \sigma}{k_z} \tag{42}$$

which is the product of vertical group velocity and the damping time. (For these waves, vertical group and phase velocities are equal and opposite.) When $\omega = 0$, equation (33) reduces to:

$$w_n \propto e^{-z/L_D} \tag{43}$$

The WKB approximation developed in Chapter II allows extension of this work to non-isothermal atmospheres, providing the scale height and vertical gradient of scale height vary slowly over a vertical wavelength. In this case, κ must be replaced by $\left(\kappa + \frac{dH_0}{dz}\right)$. This alters only equation (38), the definition of R_z, which now becomes:

$$R_z \equiv \frac{\left(\kappa + \frac{dH_0}{dz}\right)^{1/2} R_\xi\, g^{1/2}}{H_0^{1/2}\left(\sigma^2 - 4\omega^2\right)^{1/2}} \tag{44}$$

In the troposphere, $\left(\kappa + \frac{dH_0}{dz}\right) \simeq 0.1$, so that $R_z \simeq 100\ R_\xi$. Therefore, (37) corresponds to horizontal wavelengths of 1000 km, and (38) corresponds to horizontal wavelengths of 160 km. Waves longer than 1000 km will be transmitted through the troposphere with little attenuation; the stratosphere with much lower eddy viscosity will offer no damping to them, and they should reach the mesosphere. Waves shorter than 1000 km will be damped in the troposphere, and waves of 100-200 km length may be considerably distorted by viscosity.

C. Viscous Damping for the Primary Tide with a Smooth Earth

Jacob Bjerknes, (1949), suggested that ground friction might alter the phase of the tide, and represent a loss of tidal energy. The simplified analysis of the Chapter permits at least a qualitative examination of this possibility.

The solutions to a viscid atmosphere for the tidal equations contain an inhomogeneous solution and six homogeneous solutions. Three of the latter may be ruled out by the high level boundry conditions that kinetic energy of a unit column be finite, and that no mode propagate energy downward from great heights. The other three are required to cancel the three velocity components of the inhomogeneous solution at the surface of the earth. As two of these are viscous-coupled waves, even a smooth earth surface may bring about a loss of tidal energy.

The viscous modes of oscillation for waves longer than 200 km described previously have a vertical dependence:

$$e^{-(1+i)\sqrt{\frac{\sigma \pm 2\omega}{2\nu}}} \qquad (45)$$

to the first order approximation. To the same order, they have no pressure fluctuations or vertical velocities, and also:

$$u = \pm i v \qquad (46)$$

The vertical energy flux in such waves may be obtained as follows; consider the atmosphere to be divided by a level surface. The retarding force per unit area exerted on the lower section by the upper is (Lamb, 1932):

$$-\rho_0 \nu \left[\frac{\partial \vec{V}_\sigma}{\partial z}\right]_{z=z_0} = \rho_0 \sqrt{\frac{\nu(\sigma \pm 2\omega)}{2}}\,(1+i)\,\vec{V}_\sigma \qquad (47)$$

The mean work per unit area done by the lower fluid on the upper is equal to the product of the horizontal velocity and the retarding force per unit area,

averaged over a cycle, or:

$$\rho_0 \sqrt{\frac{\nu(\sigma \pm 2\omega)}{2}} \quad |\vec{V}_\sigma|^2 \tag{48}$$

when (6) is taken into account. This represents an energy flux transported upward by the wave, which is viscously dissipated at higher levels. If $\sigma = 1.5\mathrm{x}10^{-4}\ \mathrm{sec}^{-1}$, $2\omega = 1.0\mathrm{x}10^{-4}\ \mathrm{sec}^{-1}$, $\rho_0 = 1.2\ \mathrm{kg\ m}^{-3}$, and $\nu = 10\ \mathrm{m}^2\ \mathrm{sec}^{-1}$, (8) is equivalent to:

$$42.5\ |\vec{V}_\sigma|^2 \tag{49}$$

or:

$$19\ |\vec{V}_\sigma|^2 \tag{50}$$

depending on the mode excited. If a number of modes of the same vertical structure are present, their velocities may be summed vectorially, and the resultant used in this computation; it is the total resultant force and the total velocity that must be multiplied. If $|\vec{V}_\sigma|$ can be estimated, the viscous energy losses may also be obtained.

At the most, one might attribute all of the observed ground level tidal wind motions to viscous modes. These winds are $\sim$ 0.2 m sec^{-1}. In the worst case, this leads to a viscous loss of 1.7 mw m^{-2}. However, if this were the case, one would find much more variation with height of the wind amplitude and phase than is found at Terciera. It therefore seems that viscous losses do not exceed 1 mw m^{-2}, and cannot account for the downward flux of energy in the primary tidal wave.

D. Viscous Damping in the Upper Atmosphere

It is generally felt that eddy viscosity will be much smaller in the stratosphere than in the troposphere. The temperature lapse rate creates a much more stable atmosphere in which turbulence is suppressed vertically. In the mesodecline, however, stability is again relatively low, and an increased eddy viscosity might be expected. An additional source of turbulence may be active near the mesopause; internal gravity waves with large vertical shear are observed in this region, (Hines, 1961). On the basis of meteor trail and sodium vapor cloud measurements, Hines, (1963), estimates a value for ν of 100 $m^2\ sec^{-1}$ at 90 km. This is a factor of ten larger than the value assumed for the troposphere, and wavelengths of 2000-3000 km may be strongly damped.

Above 100-110 km, molecular viscosity becomes the dominant factor in viscous dissipation. Table IV shows values of the molecular kinematic viscosity as a function of height in the E-region.

Table IV. Molecular kinematic viscosity in the lower ionosphere.

Height (km)	Kinematic Viscosity ($m^2\ sec^{-1}$)	L_c (km)
100	36	8
110	300	23
120	1,700	55
130	5,400	98
140	12,000	145

These figures should not be taken to have more than order-of-magnitude significance. They are derived from Sutherland's empirical formula:

$$\nu = \frac{1.46 \times 10^{-6} \, T_0^{3/2}}{\rho_0 (T_0 + 110)} \tag{51}$$

with temperatures and densities from the ARDC 1959 model atmosphere, (Minzner, Champion, and Pond, 1959). The latter are not known accurately, and (51) has been extrapolated beyond the range for which it was derived. In addition, (51) is valid only for air of normal composition; above 100 km, oxygen dissociation and diffusive separation change the composition. However, gases of roughly comparable molecular weight do not differ by more than a factor of two in viscosity, so that the equation is at least valid to order-of-magnitude.

Table IV also shows the vertical wavelength, L_c, for which the damping time constant, $(\nu k_z^2)^{-1}$ is equal to the semidiurnal tidal period. Shorter waves would be expected to be strongly damped at the corresponding levels. At these heights, the horizontal wavelengths are about 300 times the vertical wavelengths, so that by a height of 130 km, even waves of global scale will be damped.

At these heights, wave amplitudes are large enough so that non-linear interactions occur. These interactions may also feed energy to shorter waves that are rapidly damped; this represents an additional energy sink for larger waves.

CHAPTER VI. HYDROMAGNETIC DAMPING IN THE IONOSPHERE

A. Introduction

In the ionosphere above 100 km, hydromagnetic effects play an increasingly important role in atmospheric motions. A search of the literature was undertaken to attempt to find what role such forces might play in dissipating either primary or secondary tidal waves.

This search showed considerable interest in the ionospheric interactions of wind and electric current, but primarily from the standpoint of current generation rather than wind dissipation. It is generally assumed that the winds "shall be inexorable" (Dungey, 1959). It has been pointed out qualitatively that electric fields generated in the E-region may produce motion of the gas as well as currents in the F-region, (Baker and Martyn, 1953).

The only paper discovered that dealt specifically with the dissipation of the winds is by V.P. Dokuchaev (1959). In this study, he makes a number of simplifying assumptions that allow the decoupling of the kinematic and electromagnetic equations, and thus is able to obtain a simple form of the equations of horizontal motion for the air. He obtains a damping term that arises from the transverse conductivity, and a coriolis-like term stemming from the Hall conductivity.

One of Dokuchaev's assumptions is in direct contradiction to assumptions made by a number of other workers, (Baker, 1953; Baker and Martyn, 1953; Ratcliff, 1959, 1960; Dungey, 1959). He assumes that no polarization

electric fields are set up by charge unbalance. On the other hand, Baker and Martin used a conductive sheet model of the ionosphere and concluded that polarization fields would completely inhibit the Hall current, and greatly enhance the transverse conductivity. It appears that Dokuchaev's arguments are not valid, and that polarization does play an important role. However, the results of Baker and Martyn may also be criticized; the authors were well aware of the appreciable limitations of the sheet model and worked to circumvent them. It also appears that their conclusions are valid only for an irrotational wind field.

We shall first present Dokuchaev's results, and then discuss how they might be modified through the theories of Baker and Martyn. The resulting equations of horizontal motion contain two damping terms. One, like the comparable term by Dokuchaev, is decoupled; the second stems from the polarization field, and is not. It can be shown that the second term is normally of the same order as the first, and that if it is neglecting in integrating the rate of energy dissipation over an ionospheric "sheet", the result is an over-estimate. Physical arguments may also be given to show that substitutions of the Baker-Martyn conductivities in the Dokuchaev equations leads to an over-estimate of damping losses, since all the errors tend to produce excessive transverse currents.

B. The Dokuchaev Equations

Dokuchaev begins with the equation of motion:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla P + \frac{1}{\rho c}[\vec{j}\times\vec{H_0}] - \vec{g} + 2\vec{\omega}\times\vec{V} + \nu\nabla^2 V \tag{1}$$

$\vec{V}$ = air velocity
ρ = air density
P = air pressure
ν = kinematic viscosity
$\vec{g}$ = gravitational acceleration
$\vec{\omega}$ = angular velocity of earth
$\vec{H}$ = magnetic field of earth
$\vec{j}$ = electric current density
c = speed of light

(The symbols H , ϕ and λ are used with different meanings in this chapter than they are used elsewhere, in order to agree with the original papers. As they are never used for their alternate meanings in this chapter, no confusion should result).

The standard assumptions are made that the gas is incompressible, that $\vec{H}$ varies only very slowly, and that $(\vec{V}\cdot\nabla)\vec{V}$ is negligible.

The generalized Ohm's Law is also used:

$$\vec{j} = \sigma_0[\vec{E}'\cdot\vec{h}_0]\vec{h}_0 + \sigma_1[\vec{h}_0\times\vec{E}']\times\vec{h}_0 + \sigma_2[\vec{h}_0\times\vec{E}'] \tag{2}$$

Here $\vec{h}_0$ is a unit vector in the direction of $\vec{H}$, and σ_0, σ_1, and σ_2 are respectively the longitudinal, transverse, and Hall conductivities. $\mathcal{N}$ is the electron, (and approximately the ion) density, while e, m_e, and ν_e are the electron charge, mass, and frequency of collision with neutrals, and e, m_i, and ν_i are the corresponding ion quantities. $\omega_e \equiv \frac{eH}{m_e}$ and $\omega_i \equiv \frac{eH}{m_i}$ are the electron and ion gyrofrequencies. Also the conductivities are given by:

$$\sigma_0 = \mathcal{N} e^2 \left[\frac{1}{m_e \nu_e} + \frac{1}{m_i \nu_i} \right] \tag{3}$$

$$\sigma_1 = \mathcal{N} e^2 \left[\frac{\nu_e}{m_e(\nu_e^2 + \omega_e^2)} + \frac{\nu_i}{m_i(\nu_i^2 + \omega_i^2)} \right] \tag{4}$$

$$\sigma_2 = \mathcal{N} e^2 \left[\frac{\omega_e}{m_e(\nu_e^2 + \omega_e^2)} + \frac{\omega_i}{m_i(\nu_i^2 + \omega_i^2)} \right] \tag{5}$$

Finally:

$$\vec{E}' = \vec{E} + \frac{1}{c}[\vec{V} \times \vec{H}] \tag{6}$$

The electric field, $\vec{E}$ consists of an irrotational portion $\vec{E}''$, and a non-divergent portion, $\vec{E}'''$. Since:

$$\nabla \times \vec{E}''' = -\frac{1}{c} \frac{\partial \vec{h}}{\partial t} \tag{7}$$

$\vec{h}$ being the variable portion of $\vec{H}$, and

$$|\vec{h}| \ll |\vec{H}| \tag{8}$$

Dokuchaev concludes by dimensional arguments that $|\vec{E}'''| \ll |\frac{\vec{V} \times \vec{H}}{c}|$

Dungey, (1959), reaches the same conclusion.

The irrotational, or electrostatic field is given by:

$$\nabla \cdot \vec{E}'' = 4\pi \phi \tag{9}$$

where ϕ is a net charge density, Dokuchaev argues that since the relaxation time for charge unbalance in the ionosphere is a small fraction of a second, and since the atmospheric time scale is many seconds, ϕ must be zero.

While it is true that charge unbalance sets up fields that produce neutralizing currents, and that most plasmas are quasi-neutral, other agencies may act to reinforce charge unbalance, and the resulting polarization occurs as a balancing of rates. (Note that since atmospheric motions are much slower than the times for electromagnetic adjustment, we may treat the electrical problems as steady-state.)

We shall return to this point in a moment. However, continuing Dokuchaev's arguments, we consider the Hartmann number:

$$M^2 = \frac{\sigma_1 H^2 L^2}{\nu \rho c^2} \tag{10}$$

In the E-layer, Dokuchaev takes $\sigma_1 = 4.5 \times 10^5$ /sec, $\nu\rho = 10^{-4}$ gm/sec-cm, $H = 0.5$ gauss, $L = 40$ km, and $M^2 \simeq 20$. While it might be argued that vertical dimensions for L may be somewhat smaller, the effective value of σ_1 may also be much larger. The Hartmann number stays above unity in

the F-region, according to Dokuchaev, since L also increases in this region. (Again, this is questionable.) On this basis, the viscous term can be dropped from the equation of motion.

With the above approximations the equations of motion for the u, (eastward) and v, (northward) wind components become:

$$\frac{\partial u}{\partial t} - \left(2\omega_z + \frac{\sigma_2 H H_z}{\rho c^2}\right) v + \frac{\sigma_1 H_z^2}{\rho c^2} u = -\frac{1}{\rho}\frac{\partial P}{\partial \xi} \tag{11}$$

$$\frac{\partial v}{\partial t} + \left(2\omega_z + \frac{\sigma_2 H H_z}{\rho c^2}\right) u + \frac{\sigma_1 H_z^2}{\rho c^2} = -\frac{1}{\rho}\frac{\partial P}{\partial \eta} \tag{12}$$

Here:

$$H = \sqrt{H_\eta^2 + H_z^2} \tag{13}$$

and H_η and H_z are the northward and vertical components of $\vec{H}$, and ω_z the vertical component of $\vec{\omega}$. If:

$$\lambda = \frac{\sigma_1 H_z^2}{\rho c^2} \tag{14}$$

$$\Omega = 2\omega_z + \frac{\sigma_2 H H_z}{\rho c^2} \tag{15}$$

then equations (11) and (12) become:

$$\frac{\partial u}{\partial t} - \Omega v + \lambda u = -\frac{1}{\rho}\frac{\partial P}{\partial \xi} \tag{16}$$

$$\frac{\partial v}{\partial t} + \Omega u + \lambda u = -\frac{1}{\rho}\frac{\partial P}{\partial \eta} \tag{17}$$

If we multiply by u and v respectively, and add, we obtain the horizontal kinetic energy equation:

$$\frac{\partial}{\partial t}\left(\frac{u^2+v^2}{2}\right) + 2\lambda\left(\frac{u^2+v^2}{2}\right) = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial \xi} + v\frac{\partial p}{\partial \eta}\right) \tag{18}$$

The second term represents a dissipation with a time constant $(2\lambda)^{-1}$. In a tidal system, the long time average of kinetic energy is a constant. In this simplified equation, the dissipation must be balanced, on the average, by work done by pressure forces as represented by the right hand side of (18). Had the nonlinear terms been left in, advection of horizontal kinetic energy would also have been possible. The damping coefficient λ is given as a function of height in Figure 11. These data were based on conductivities by Baker and Martyn, (1953), and densities from the 1959 ARDC Model atmosphere, (Minzner, Champion, and Pond, 1959). (Strictly speaking, the same densities should have been used in computing the conductivities. However, other parameters are even less well known).

The Hall conductivity may markedly alter the coriolis parameter, as may also be shown from a plot of $\frac{\sigma_2 H H_3}{\rho c^2}$ as a function of height. This effect depends on the assumption of charge neutrality, however.

Now return to the original critical assumption of charge neutrality. As a simplified model, consider an infinite flat geometry with a normal magnetic field and no initial polarization. Then:

$$j_\xi = \frac{\sigma_1 v H}{c} + \frac{\sigma_2 u H}{c} \tag{19}$$

$$j_\eta = -\frac{\sigma_1 u H}{c} + \frac{\sigma_2 v H}{c} \tag{20}$$

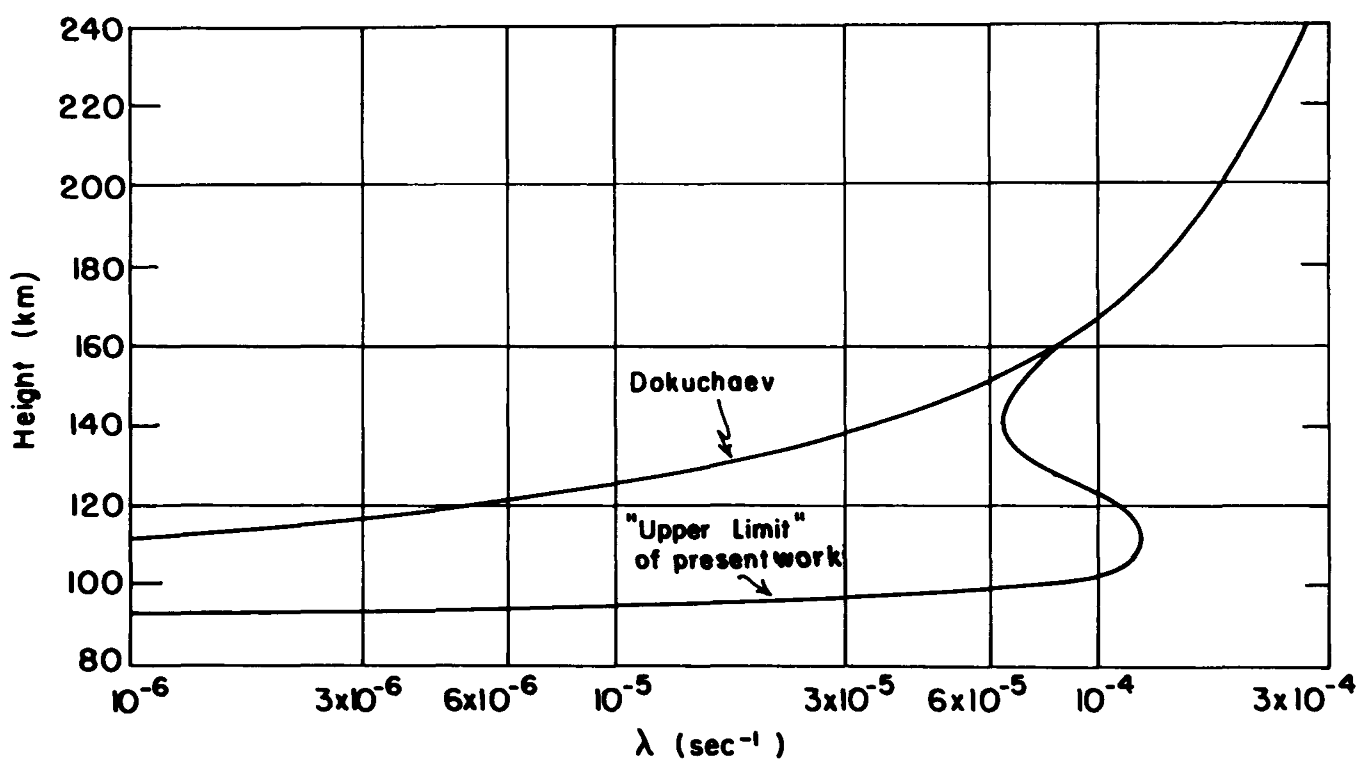

Figure 11. Hydromagnetic damping coefficient as a function of height in the ionosphere.

and the vertical current is zero, providing there is no vertical velocity. Therefore:

$$\nabla \cdot \vec{j} = \frac{H}{c}\left[\sigma_1\left(\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) + \sigma_2\left(\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta}\right)\right] \tag{21}$$

Were the gas incompressible, and in two-dimensional flow, $\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0$ but even in this case the flow must also be irrotational to avoid the creation of a polarization field. Dokuchaev's assumption requires a very strong restriction on the nature of the velocity field, since without polarization, in general, the current is divergent.

C. The Results of Baker and Martyn

Before embarking on a quantitative discussion, it is worth while to gain some physical insight through an argument presented by Baker, (1953). Consider an arbitrary infinite sheet having a transverse conductivity σ_1, and a Hall conductivity σ_2. Let an electric field be applied by means of arbitrary electrodes. The Hall current then flows along equipotential lines and the transverse current along field lines. Since the equipotential curves are closed, the Hall current causes no polarization.

If the sheet is now terminated by a finite boundary, some equipotential lines will be intercepted, and the Hall current will cause a charge to build up on the boundary, with a resulting polarization field. No current will flow between pairs of points on the boundary. If the electrodes lie on the boundary, all of the equipotential lines are intercepted, and no Hall

current at all will flow. The resulting polarization field will also react through the Hall conductivity to cause a current to flow along the original field lines. The final result is as though the Hall conductivity were zero and the transverse conductivity were:

$$\sigma_3 = \sigma_1 + \frac{\sigma_2^2}{\sigma_1} \tag{22}$$

Baker then considers an arbitrary surface, and under a set of assumptions, shows that the Hall current is completely cancelled, and that the effective transverse conductivity is given by (22). The assumptions are:

1. The transverse and Hall conductivities are independent of direction. Physically, this corresponds to a normal magnetic field.
2. Both conductivities are uniform over the surface.
3. The surface may be treated as infinitely thin.
4. The wind field may be represented by a scalar velocity potential, ψ , and lies within the sheet.

The first assumption is normally met within a factor of two, except near the equator. The second is quite inexact, as the conductivities vary by almost an order of magnitude from night to day. Some, but not all of the problems of the third assumption can be avoided by using vertically integrated conductivities. The final assumption restricts us to irrotational motion. Under these assumptions, the top and bottom of the sheet are polarized so as to oppose any vertical current.

It appears that the fourth assumption is very restrictive, and automatically gives rise to the cancellation of the Hall current. If the wind field is described by a velocity potential such that:

$$\vec{V} = -\nabla \psi \tag{23}$$

and H is normal to the surface, the induced e.m.f., $[\vec{V} \times \vec{H}]/c$ lies along equipotentials. There are no lateral boundries on the spherical shell to intercept these equipotentials, the form sets of closed curves on the surface, along which the transverse current is free to flow.

The Hall current must flow along lines normal to these closed curves, the "field lines" of ψ , and these lines must necessarily converge to points. If a Hall current did flow, it would be convergent, and this is prohibited in the steady state. Therefore a polarization field must be set up to oppose the Hall current at all points. This polarization field must be such that by itself it would cause a direct, or transverse current equal and opposite to the induced Hall current. (It cannot cancel though its own Hall current, as this would imply the integral of grad ϕ around a closed loop was not zero.) The polarization field's Hall current will add to the induced transverse current, as noted before.

An unbounded vector field, such as the velocity, may be uniquely resolved into two component fields, one irrotational, and the other solenoidal, (c.f. Newell, 1955). The irrotational portion may be represented by a scalar potential. If the field is two-dimensional the solenoidal portion may be represented by a stream function, (c.f. Milne-Thompson, 1955).

The conclusions drawn by Baker are valid only for the first portion of the motion.

Now consider the second case, representable by a stream function Φ , such that:

$$\vec{V} = \vec{n} \times \nabla \Phi \qquad (\vec{n} \text{ normal to surface}) \qquad (24)$$

The velocity now lies along closed curves, while the induced e.m.f. lies along normals to these curves. The transverse current now would be convergent, and must be completely cancelled by the polarization field. But in this case, the polarization and induced fields are everywhere equal and opposite, so that the total field is zero, and no current at all will flow. (See section D.)

Thus the irrotational velocity field will have enhanced damping from its own currents, while the solenoidal velocity field produces no currents. It will interact with the other field's currents in an undefined manner, locally producing either positive or negative damping.

One may try to obtain some physical insight as to the effects of the other assumptions. In the irrotational case that Baker treats, a direct current flows in closed loops. We can envision a tube of current, along which there are both induced and polarization e.m.f.s., the polarization being so distributed to keep the current constant, despite changes along the length of the induced e.m.f., or the conductivity per unit length of the tube. The latter will vary if the tube cross-section varies. One can also let it vary through change in σ_3 . A varying conductivity will

thus create a different polarization field, but does not qualitatively change the mechanisms involved. It is strongly suspected that a varying conductivity could be incorporated in Baker's equations by replacing $\vec{H}$ and ϕ by $\sigma_3 \vec{H}$ and $\sigma_3 \phi$ as variables. Of course, since no current flows in the case of a solenoidal wind, the conductivity is irrelevant.

If the magnetic field is not perpendicular to the surface, the induced e.m.f. will have a normal component. As noted, vertical polarization will cancel this component; the remainder of the induced e.m.f. is equivalent to that which would arise from the vertical component of the magnetic field.

If the wind field is vertically uniform, a vertical variation in conductivity may be treated by the use of vertically integrated conductivities. Baker, and Baker and Martyn have followed this approach. We shall not repeat their arguments, but simply notice that physically this allows a Hall current to flow by closing a vertical loop. This reduces the enhancement of direct conductivity. Baker and Martyn suggest that such leakage may be cancelled by the induction of motions in the short circuiting region. This is the dynamo theory in which the E layer dynamo drives the F layer motor; when the motor is in motion, it produces a back e.m.f. that reduces current flow.

D. Currents Induced by a Solenoidal Velocity Field

We shall follow the notation used by Baker in his dynamo theory for an arbitrary surface having two orthogonal coordinates u and v. The elements of length are $h_1\,du$ and $h_2\,dv$. Assume a magnetic field having at any point a normal component H_N. As noted previously, tangential components of $\vec{H}$ have their effects cancelled by vertical polarization. Let $\vec{E}^\circ$ be the total electric field, with components E_u° and E_v° in the plane. The transverse conductivity σ_{uu} and the Hall conductivity σ_{uv} will be assumed everywhere uniform and independent of direction.

As the current system must close, a current function R exists such that the current densities are given by:

$$I_u = \frac{1}{h_2}\frac{\partial R}{\partial v} \qquad I_v = \frac{-1}{h_1}\frac{\partial R}{\partial u} \tag{25}$$

From the generalized Ohm's Law, the equations for I_u and I_v become:

$$\frac{1}{h_2}\frac{\partial R}{\partial v} = \sigma_{uu}E_u^\circ + \sigma_{uv}E_v^\circ \tag{26}$$

$$-\frac{1}{h_1}\frac{\partial R}{\partial u} = -\sigma_{uv}E_u^\circ + \sigma_{uu}E_v^\circ \tag{27}$$

We now eliminate R.

$$\frac{1}{h_1 h_2}\frac{\partial^2 R}{\partial u\,\partial v} - \frac{1}{h_1 h_2^2}\frac{\partial h_2}{\partial u}\frac{\partial R}{\partial v} = \frac{\sigma_{uu}}{h_1}\frac{\partial E_u^\circ}{\partial u} + \frac{\sigma_{uv}}{h_1}\frac{\partial E_v^\circ}{\partial u} \tag{28}$$

$$\frac{-1}{h_1 h_2}\frac{\partial^2 R}{\partial u\,\partial v} + \frac{1}{h_1^2 h_2}\frac{\partial h_1}{\partial v}\frac{\partial R}{\partial u} = -\frac{\sigma_{uv}}{h_2}\frac{\partial E_u^\circ}{\partial v} + \frac{\sigma_{uu}\,\partial E_v^\circ}{v} \tag{29}$$

Adding (28) and (29) and substituting from (26) and (27)

$$-\frac{1}{h_1 h_2}\left[\sigma_{uu} E_u^\circ + \sigma_{uv} E_v^\circ\right]\frac{\partial h_2}{\partial u} - \frac{1}{h_1 h_2}\left[\sigma_{uu} E_v^\circ - \sigma_{uv} E_u^\circ\right]\frac{\partial h_1}{\partial v} \tag{30}$$

$$-\frac{\sigma_{uu}}{h_1}\frac{\partial E_u^\circ}{\partial u} - \frac{\sigma_{uv}}{h_1}\frac{\partial E_v^\circ}{\partial u} + \frac{\sigma_{uv}}{h_2}\frac{\partial E_u^\circ}{\partial v} - \frac{\sigma_{uu}}{h_2}\frac{\partial E_v^\circ}{\partial u} = 0$$

or:

$$\frac{\sigma_{uu}}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 E_u^\circ) + \frac{\partial}{\partial v}(h_1 E_v^\circ)\right]$$

$$+ \frac{\sigma_{uv}}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 E_v^\circ) - \frac{\partial}{\partial v}(h_1 E_u^\circ)\right] = 0 \tag{31}$$

or:

$$\sigma_{uu}\,\nabla\cdot\vec{E}^\circ + \sigma_{uv}\,\nabla\times\vec{E}^\circ = 0 \tag{32}$$

$\vec{E}^\circ$ consists of solenoidal and irrotational components. The polarization field is given by the gradient of a scalar potential, thus being irrotational.

Now consider a solenoidal velocity field, given by:

$$\vec{V} = \vec{n}\times\nabla\Phi \tag{33}$$

where $\vec{n}$ is a unit vector normal to the surface. The induced e.m.f. is:

$$\frac{\vec{V}\times\vec{H}_1}{c} = -\frac{H_N\nabla\Phi}{c} \tag{34}$$

The curl of the induced field is:

$$-\frac{H_N}{c}\,\nabla\times\nabla\Phi = 0 \tag{35}$$

so that in this case, the induced field is also irrotational. Therefore,

is irrotational, and $\nabla \times \vec{E}^{\circ}$ is zero. But from (32), $\nabla \cdot \vec{E}^{\circ}$ must then also be zero. Since $\vec{E}^{\circ}$ contains neither non-divergent, (rotational) nor irrotational components, it must be zero. If that is so, no currents will be generated.

E. An Upper Limit to Hydromagnetic Damping

If daytime values of σ_3 are substituted into equation (14), in place of σ_1 , the effects of polarization are taken into consideration. However, at least three important sources of error remain. The damping is valid only for the irrotational portion of the wind, the effects of vertical current closure have been ignored, and the day-night variation of conductivity has not been taken into account. From the previous discussion, it appears that all three errors act in the direction to produce an overestimate of the viscous damping, so that a reasonable upper limit to the magnitude of hydromagnetic damping may be obtained.

Figure 11 shows λ as a function of height, using σ_3 ; for comparison, Dokuchaev's values with σ_1 are also shown. If λ approaches the tidal frequency, hydromagnetic damping becomes important. It is possible, though of course not proven, that this may occur above 100 km. As has been noted, molecular viscosity rapidly becomes a dominating factor at these heights, and viscous damping will likely exceed hydromagnetic damping for vertical wavelengths shorter than 30 km in the E region, and should certainly predominate for any waves which can propagate to 150 km.

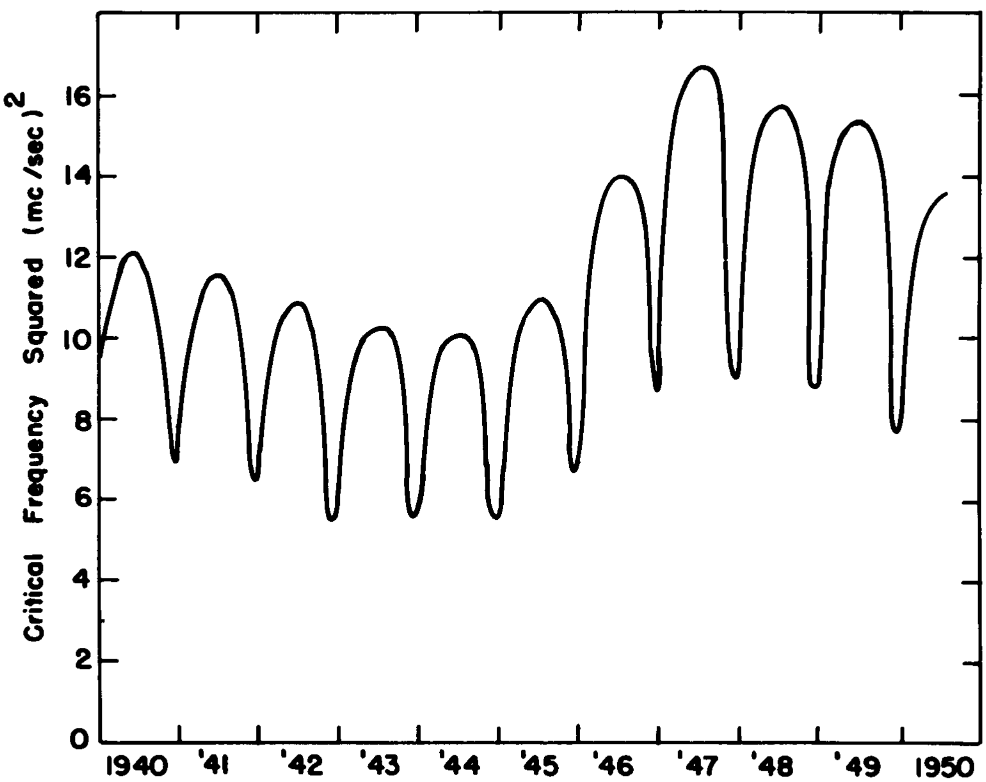

Figure 12. Annual variation of E-layer critical frequency squared, at midday, for Slough, England. (Mitra, 1951.)

If hydromagnetic damping is important to the primary tide at these high levels, long-term variations in the mean conductivity should be reflected in the tide. The conductivity has a seasonal variation, as shown in Figure 12. The square of the noon-day critical frequency, (a measure of electron density) is plotted as a function of season, at Slough, England. Values are for the E-region. Conductivity is reduced considerably near the winter solstice. The tidal motions at 80-100 km, (see Chapter VII) are at a maximum at this time, which might be expected if damping losses were reduced. However, there is usually an additional maximum in September that cannot be explained on this basis.

Figure 12 also suggests that an eleven year period be looked for in the tides at these heights, if hydromagnetic damping is important. The existing data at these levels covers only a four year period, and no identification with such a period can be made; there is a small trend toward decreasing amplitudes in corresponding months of successive years, during a time span, (1953-1957) of increasing sunspot activity. These observations offer tantalizing encouragement, but no proof, whatever.

There is also the possibility, which will not be taken up here, that gravity waves may couple energy to hydromagnetic waves, and vice-versa.

CHAPTER VII. EFFECTS OF EDDY CONVECTION

In Chapter IV it was shown that differential heating, through a positive correlation between the tidal temperature and water vapor absorption of solar radiation leads to a generation of tidal energy. It is equally possible that a negative correlation of the temperature fluctuation with diabatic heating from another source might lead to the destruction of tidal energy. One possible source is eddy convective heating from the ground. If such an effect were operative it would result in an energy sink close to the ground; it would then offer an explanation for the downward flux of tidal energy. This mechanism thus requires further exploration.

From the results of Chapter IV, the mean rate of generation of tidal available potential energy per unit volume is:

$$\frac{\Gamma_d \, \rho_0}{2(\Gamma_d - \Gamma_0) T_0} \, \mathcal{R}e \left[T_\sigma' \, J_\sigma^{*\prime} \right] \qquad (1)$$

where the primes refer to Lagrangrian parameters, Γ_d is the dry adiabatic lapse rate, and Γ_0 the actual thermal lapse rate. (As has been done in earlier chapters, T_0 will be taken from the ARDC 1959 Model Atmosphere.) It has also been shown that (1) may be written approximately as:

$$\simeq \frac{\kappa \, \Gamma_d \, \rho_0}{2(\Gamma_d - \Gamma_0) P_0} \left[\mathcal{R}e \left[P_\sigma \, J_\sigma^{*} \right] + \mathcal{R}e \left(\frac{w_\sigma}{i\sigma} \frac{\partial P_0}{\partial z} J_\sigma^{*} \right) \right] \qquad (2)$$

Like the effects of eddy viscosity, the amplitude of the semidiurnal eddy conductivity heat flow decreases rapidly with height, damping out within the first few hundred meters above the ground. Over this range, ρ_0 and P_0 may be taken as approximately constant. From the observational data, Γ_0 also changes comparatively little over such a height range and hence also will be taken to be constant to the crude approximations made here. These approximations simplify the evaluation of the first half of expression (2).

If the term in W_σ is neglected, the total rate of energy generation by convective heating, (hopefully negative), between the ground and some level z_0 at which convective heating is negligible is·

$$W_\sigma \simeq \frac{\kappa \Gamma_d \rho_0}{2(\Gamma_d - \Gamma_0) P_0} \int_0^{z_0} \mathcal{R}e\left[P_\sigma J_\sigma^* \right] dz' \tag{3}$$

or:

$$W_\sigma \simeq \frac{\kappa \Gamma_d \rho_0 |P_\sigma|}{2(\Gamma_d - \Gamma_0) P_0} \mathcal{R}e\left[\frac{P_\sigma}{|P_\sigma|} C_\sigma^* \right] \tag{4}$$

Here C_σ is the upward convective heating flux at ground level. Horizontal convergence of diabatic heating is small in this layer. If numerical values are substituted equation (4) becomes:

$$W_\sigma \simeq 2.5 \times 10^{-4} \mathcal{R}e\left[\frac{P_\sigma}{|P_\sigma|} C_\sigma^* \right] \quad w/m^2 \tag{5}$$

If this energy sink is to balance the downward flux of 7×10^{-3} w./m.^{2}, the component of C_σ <u>out of phase</u> with P_σ must be about 30 w./m.^{2}.

The one set of convective heat flux data known to the author that may be resolved into its periodic components has been compiled by Lettau, (1949), for measurements in the Gobi Desert in June. The amplitude of the semidiurnal component was 60 w./m.^{2}. Desert conditions in June are more representative of extreme rather than mean global fluxes. The flux over oceans would be expected to be much smaller, for example. A mean convective flux of 30 w./m.^{2} at the latitude of Terciera is not out of the question, however.

The phase of the Gobi data is such that the semidiurnal flux reaches a maximum at one o'clock. This is almost exactly in time quadrature with the tidal pressure. Unless the phase of the flux at other points on the globe is retarded by several hours from this value the convective heating will not serve as an energy sink.

Computing the influence of the second term in expression (2) is more difficult, since W_σ may vary considerably. In fact, if the tidal-terrain effect is dismissed, it must go to zero at the ground. An upper limit to the magnitude of the term might reasonably be obtained by taking W_σ to have a magnitude equal to the value computed in Chapter IV and to be in phase with J_σ at all heights. The contribution of this component would then be:

$$W_\sigma = 1.2 \times 10^{-4} |C_\sigma| \quad w/m^2 \tag{6}$$

Under these most ideal of situations a magnitude of $|C_\sigma| = 60$ w./m.^{2} would be required. If W_σ goes to zero at the ground or has less than optimum phase, a still greater flux would be required. In fact, if W_σ maintained the phase and amplitude computed in Chapter IV and the vertical flux at the Gobi Desert phase, tidal energy would be generated rather than lost.

A more convincing argument against the importance of eddy convection has already been presented in part in Chapter III. If the energy flux goes to zero at the ground, then the component of W_σ out of phase with P_σ must go to zero also. In this case the change in W_n must occur over a short height. But this cannot be accomplished without corresponding abrupt changes in P_n , which are not observed in the tidal data. From equation (23) of Chapter III:

$$W_n \propto \frac{dy_n}{dx} - \left(\frac{H_0}{h_n} - \frac{1}{2}\right) y_n \qquad (7)$$

$$\simeq \frac{dy_n}{dx} + \frac{1}{2} y_n$$

in the case of the $\Theta^2_{2,2}$ mode of oscillation, while:

$$P_n \propto \frac{dy_n}{dx} - \frac{1}{2} y_n \qquad (8)$$

If abrupt changes in W_n occur over distances over which x changes by a small fraction, (say 1 km,) these changes must be reflected primarily in changes in $\frac{dy_n}{dx}$. But if the changes in $\frac{dy_n}{dx}$ are more pronounced than the changes in y_n , P_n must also change. Observationally, the change in P_n is small over these heights. It is still possible that convective heating plays a role in smaller scale perturbations in the tidal amplitude and phase, however.

CHAPTER VIII. MERIDIONAL TRANSPORT OF TIDAL ENERGY

A. Theoretical Considerations

Inspection of the results of Chapter II shows that for a single mode of oscillation, the meridional component of the tidal wind is in time quadrature with the pressure. Equation (II-20) states that:

$$V_n = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{\rho_0}\frac{\partial P_n}{\partial \eta} - \frac{2\omega}{\rho_0}\frac{\partial P_n}{\partial \xi}\right] \tag{1}$$

for a planar geometry, while for a spherical geometry, (II-47) states:

$$V_n = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{a\rho_0}\frac{\partial P_n}{\partial \theta} - \frac{2\omega}{a\rho_0\cos\theta}\frac{\partial P_n}{\partial \phi}\right] \tag{2}$$

If only a single mode of oscillation exists, the horizontal variations of all tidal parameters may be resolved into separate zonal and meridional variations. Thus along any line of constant longitude, P_n has constant phase, and $\frac{\partial P_n}{\partial \eta}$, (or $\frac{\partial P_n}{\partial \theta}$, as the case may be) has the same phase as P_n. The zonal dependence of P_n is of the form $e^{i k_x \xi}$, (or $e^{is\phi}$,) so that $\frac{\partial P_n}{\partial \xi}$, (or $\frac{\partial P_n}{\partial \phi}$) is proportional to iP_n. It follows that V_n is in phase quadrature with P_n, and:

$$\frac{1}{2}\,\mathrm{Re}\left[V_n^{*} P_n\right] = 0 \tag{3}$$

There is no mean meridional flux of tidal wave energy.

There are circumstances under which there may be a mean flux in the meridional direction, however, Two modes may combine in such a manner as to produce a net transport. Let V_1 and P_1 be associated with one mode of oscillation, and V_2 and P_2 with a second at some given point. If:

$$\frac{|P_1|}{|V_1|} \neq \frac{|P_2|}{|V_2|} \tag{4}$$

and the waves are not in phase, then $P_1 + P_2$ need not be in quadrature with $V_1 + V_2$, and there can be a mean meridional energy transport.

If the two modes have different zonal wave numbers, simple inspection indicates that their relative phases vary through an integral number of revolutions around a latitude circle. For every point on this circle where the mean meridional transport takes on a given value, there will be a corresponding point where the relative phase is reversed and the transport will be in the opposite direction. Thus there will be no energy flow when integrated over the entire latitude circle.

On the other hand, if the two waves have the same zonal, though different meridional dependence, their relative phase will be the same at all points on a latitude circle, and there will be a net meridional transport.

As this energy flux arises from cross terms:

$$\frac{1}{2} \mathcal{R}e \left(V_2^* P_1 + V_1^* P_2 \right) \tag{5}$$

it represents a non-linear interaction between the two modes, and an inter-modal transfer of energy. It is not necessary to resort to a solution of the non-linear equations to gain some further insight, however. The following analysis will be done in terms of the rotating planar model, though the extension to the spherical case is obvious. Let the total tidal pressure be written as:

$$P_\sigma = \mathcal{P}_\sigma(\eta,\zeta)\, e^{i[\sigma x + k_\xi \xi + \delta(\eta,\zeta)]} \tag{6}$$

where $\mathcal{P}_\sigma$ and δ are real variables. Then from equations (II-20):

$$u_\sigma = \frac{P_\sigma}{\rho_0(\sigma^2-4\omega^2)}\left[-k_\xi\sigma + 2\omega\frac{\partial \ln \mathcal{P}_\sigma}{\partial \eta} + 2i\omega\frac{\partial \delta}{\partial \eta}\right] \tag{7}$$

$$v_\sigma = \frac{P_\sigma}{\rho_0(\sigma^2-4\omega^2)}\left[-2i\omega k_\xi + i\sigma\frac{\partial \ln \mathcal{P}_\sigma}{\partial \eta} - \sigma\frac{\partial \delta}{\partial \eta}\right] \tag{8}$$

The mean zonal flux of wave energy is:

$$\frac{1}{2}\mathrm{Re}[u_\sigma^* P_\sigma] = \frac{-\sigma\mathcal{P}_\sigma^2}{2\rho_0(\sigma^2-4\omega^2)}\left[k_\xi - \frac{2\omega}{\sigma}\frac{\partial \ln \mathcal{P}_\sigma}{\partial \eta}\right] \tag{9}$$

and the mean meridional flux is:

$$\frac{1}{2}\mathrm{Re}[v_\sigma^* P_\sigma] = \frac{-\sigma\mathcal{P}_\sigma^2}{2\rho_0(\sigma^2-4\omega^2)}\frac{\partial \delta}{\partial \eta} \tag{10}$$

The meridional flux is thus associated with a meridional shift in phase of the tidal pressure.

The shifts in phase of u_σ and v_σ relative to p_σ have an additional effect, producing terms that enter into the kinetic energy budget. The mean rates of conversion of zonal and meridional kinetic energy into potential and internal energy are given by:

$$\frac{1}{2} Re\left[u_\sigma^* \frac{\partial p_\sigma}{\partial \xi}\right] = \frac{P_\sigma^2 \omega k_\xi}{\rho_0 (\sigma^2 - 4\omega^2)} \frac{\partial \delta}{\partial \eta} \quad w/m^3 \tag{11}$$

and:

$$\frac{1}{2} Re\left[v_\sigma^* \frac{\partial p_\sigma}{\partial \eta}\right] = \frac{-P_\sigma^2 \omega k_\xi}{\rho_0 (\sigma^2 - 4\omega^2)} \frac{\partial \delta}{\partial \eta} \tag{12}$$

respectively. Thus for $\frac{\partial \delta}{\partial \eta} < 0$, corresponding to a mean northward energy flux, there is a gain of zoral kinetic energy and a loss of meridional kinetic energy. There is also a term, (Saltzman, 1955):

$$\frac{1}{2} Re\left[2\rho_0 \omega u_\sigma^* v_\sigma\right] = \frac{-P_\sigma^2 \omega k_\xi}{\rho_0 (\sigma^2 - 4\omega^2)} \frac{\partial \delta}{\partial \eta} \tag{13}$$

representing the mean conversion from zonal to meridional kinetic energy through the coriolis deflection, so that the horizontal kinetic energy budget is balanced in parts as well as in the whole.

The product:

$$\frac{1}{2} \mathcal{R}e\left[\rho_0 u_\sigma^* V_\sigma\right] = \frac{P_\sigma^2 R_\xi}{2\rho_0(\sigma^2 - 4\omega^2)} \frac{\partial \delta}{\partial \eta} \tag{14}$$

represents a meridional transport of zonal, or angular momentum. It is in the opposite direction to the meridional energy flux.

It is also possible to write the equivalent equations in the more general case, when waves of different zonal wave numbers are present. In this case, $\frac{\partial P_\sigma}{\partial \xi}$ is not generally proportional to $i P_\sigma$. Therefore, one must write:

$$\frac{1}{2} \mathcal{R}e\left[u_\sigma^* \frac{\partial P_\sigma}{\partial \xi}\right] = \frac{\omega}{\rho_0(\sigma^2 - 4\omega^2)} \mathcal{R}e\left[\frac{\partial P_\sigma^*}{\partial \eta} \cdot \frac{\partial P_\sigma}{\partial \xi}\right] \tag{15}$$

$$\frac{1}{2} \mathcal{R}e\left[V_\sigma^* \frac{\partial P}{\partial \eta}\right] = \frac{-\omega}{\rho_0(\sigma^2 - 4\omega^2)} \mathcal{R}e\left[\frac{\partial P_\sigma^*}{\partial \xi} \cdot \frac{\partial P_\sigma}{\partial \eta}\right] \tag{16}$$

$$= \frac{-\omega}{\rho_0(\sigma^2 - 4\omega^2)} \mathcal{R}e\left[\frac{\partial P_\sigma^*}{\partial \eta} \cdot \frac{\partial P_\sigma}{\partial \eta}\right]$$

$$\frac{1}{2} \mathcal{R}e\left[V_\sigma^* P_\sigma\right] = \frac{-\omega}{\rho_0(\sigma^2 - 4\omega^2)} \mathcal{R}e\left[\left(\frac{\partial P_\sigma^*}{\partial \xi} - \frac{i\sigma}{2\omega} \frac{\partial P_\sigma^*}{\partial \eta}\right) \cdot P_\sigma\right] \tag{17}$$

and:

$$\frac{1}{2} \mathcal{R}e\left[\rho_0 u_\sigma^* V_\sigma\right] = \frac{1}{2\rho_0(\sigma^2 - 4\omega^2)} \mathcal{R}e\left[\frac{\partial P_\sigma^*}{\partial \eta} \cdot \frac{\partial P_\sigma}{\partial \xi}\right] \tag{18}$$

The kinetic energy generation remains balanced, but the meridional energy flux is not as simply related to the energy conversion terms. The

transport of angular momentum is no longer a contribution to the mean zonal momentum, but may represent additions to time-invariant waves.

One possible cause for a phase shift with latitude in the pressure wave may be found in the tidal-terrain interaction discussed in Chapter III. The amplitudes of both the heating function and the observed tidal pressure decrease with increasing latitude. It was shown that the generation of tidal energy depends on the correlation of these parameters, and so also drops off rapidly at high latitudes.

The horizontal winds are proportional to $(\sigma^2 - 4\omega^2 \sin^2\theta)^{-1}$, and to the gradient of pressure, the eastward component of which increases relative to the pressure itself at higher latitude. The secondary wave losses, which are proportional to the square of the horizontal velocity, will thus decrease less rapidly than the ability to generate tidal energy. This will act to distort oscillational modes, and may "encourage" a northward energy transport.

Figure 3 may be used to demonstrate this point at low levels. If at higher latitudes, the horizontal winds are greater for a given pressure, the downward flow of tidal energy must also be greater. Therefore, the magnitude of w_b must be increased, relative to that of the total pressure, $P_a + P_b$. But since w_b is also proportional to P_b , the phase lag of the ground level pressure wave must increase at higher latitudes. This decrease in phase angle with latitude corresponds to a northward transport of tidal energy. Of course, changes in terrain will also have effects, of a less readily predicted nature.

It is also possible that viscous or non-linear terms that have been omitted from the equations of motion produce phase shifts and resulting meridional transport. From the results of Chapter III, the vertical portion of eddy viscosity should have little effect on the tide; the effect of large scale horizontal eddies may still be an open question. It will be shown in Chapter X that the non-linear interaction with such waves is probably small.

B. Observed Energy Transport - Terciera

The data from Terciera, (Harris, Finger, and Teweles, 1962), provide information on u_{σ} and v_{σ} , as well as the pressure of the semidiurnal tide. The meridional energy flux may thus be computed. However, the wind data is less accurate than the pressure data, and produces rather scattered results. (Note that Table I of that reference was inadvertantly computed for Greenwich, rather than local time, and that therefore, corrections of +27 degrees and +54 degrees must be added to the diurnal and semidiurnal wind phases, respectively.)

The meridional energy fluxes obtained from this data are shown in Figure 13. Below 500 mb, and above 200 mb, the flux is northward. In the intervening region it is southward. The meridional component of the tidal wind behaves abnormally in this middle region. At greater and lesser heights, it is equal to or slightly greater than the zonal component, as theory predicts. In the intervening region it drops to a third the magnitude of the zonal component. The cause for this is not understood. Presuming the effect is real, some form of reflection at the tropopause

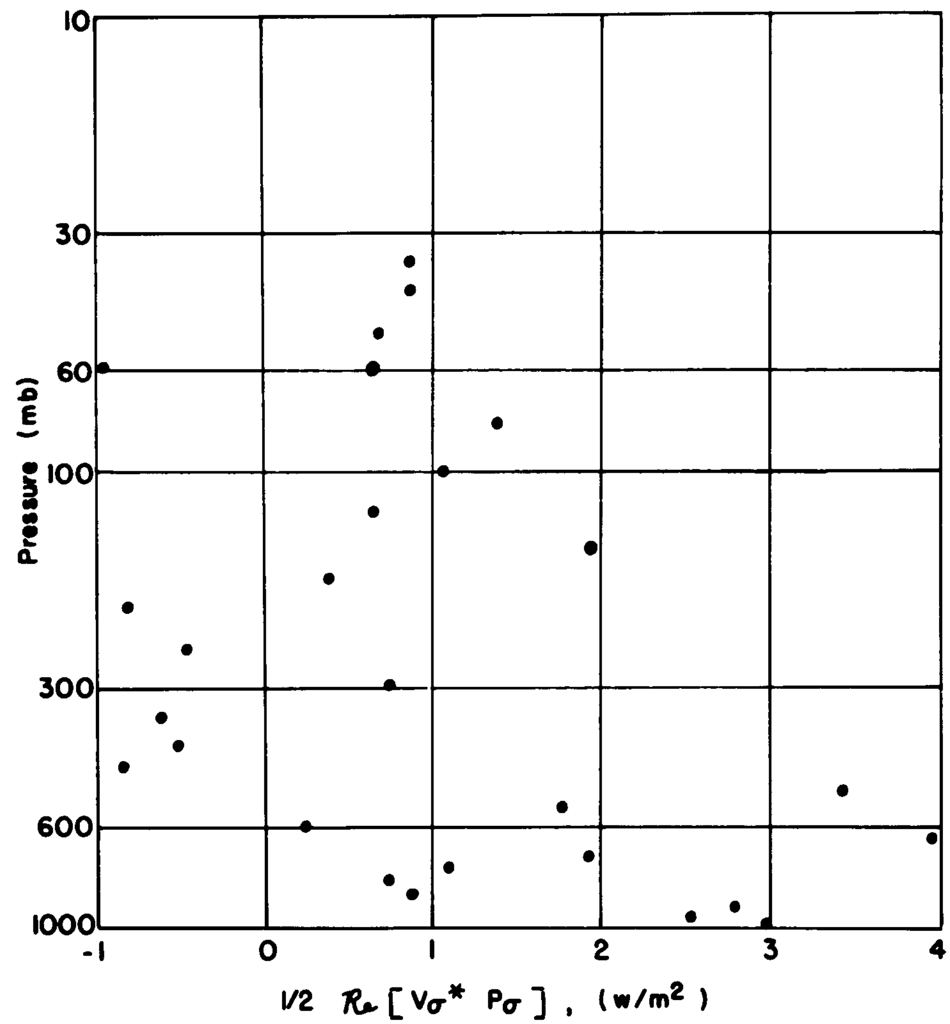

Figure 13. Meridional energy flux in the semidiurnal tide - Terciera.

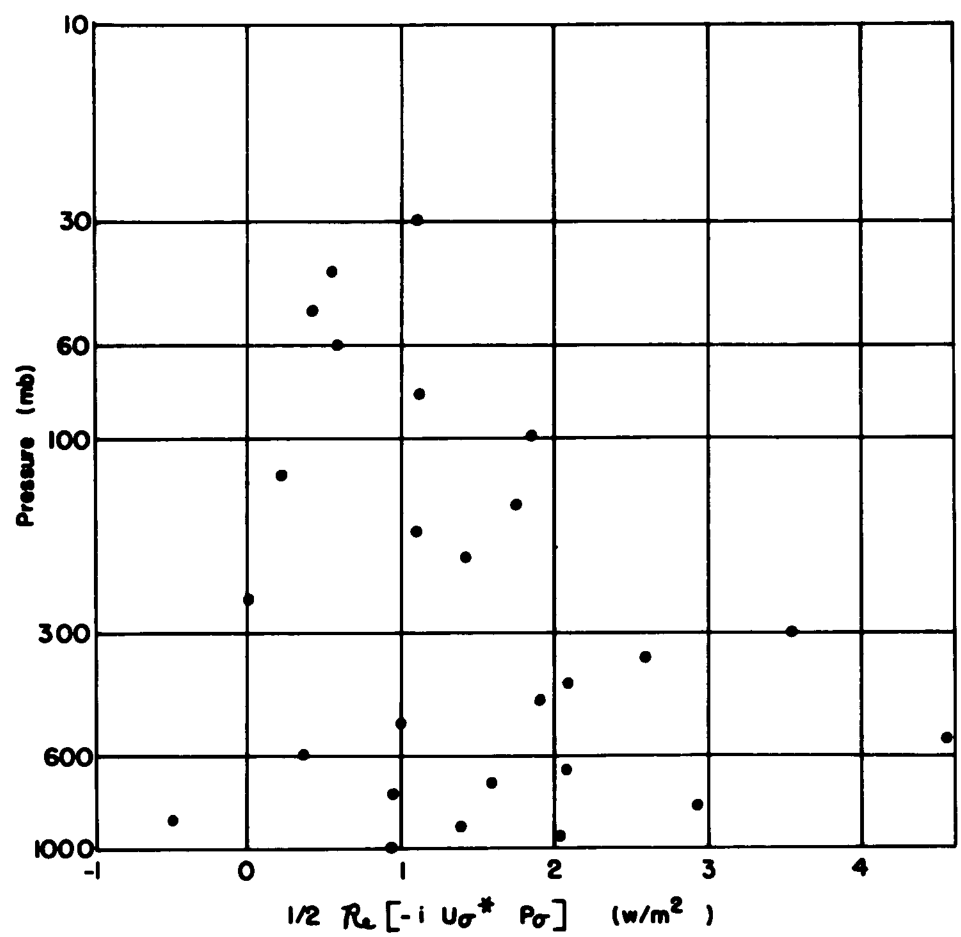

Figure 14. $\frac{1}{2} Re[-i\, U_\sigma^* P_\sigma]$ as a function of height for the semidiurnal tide - Terciera.

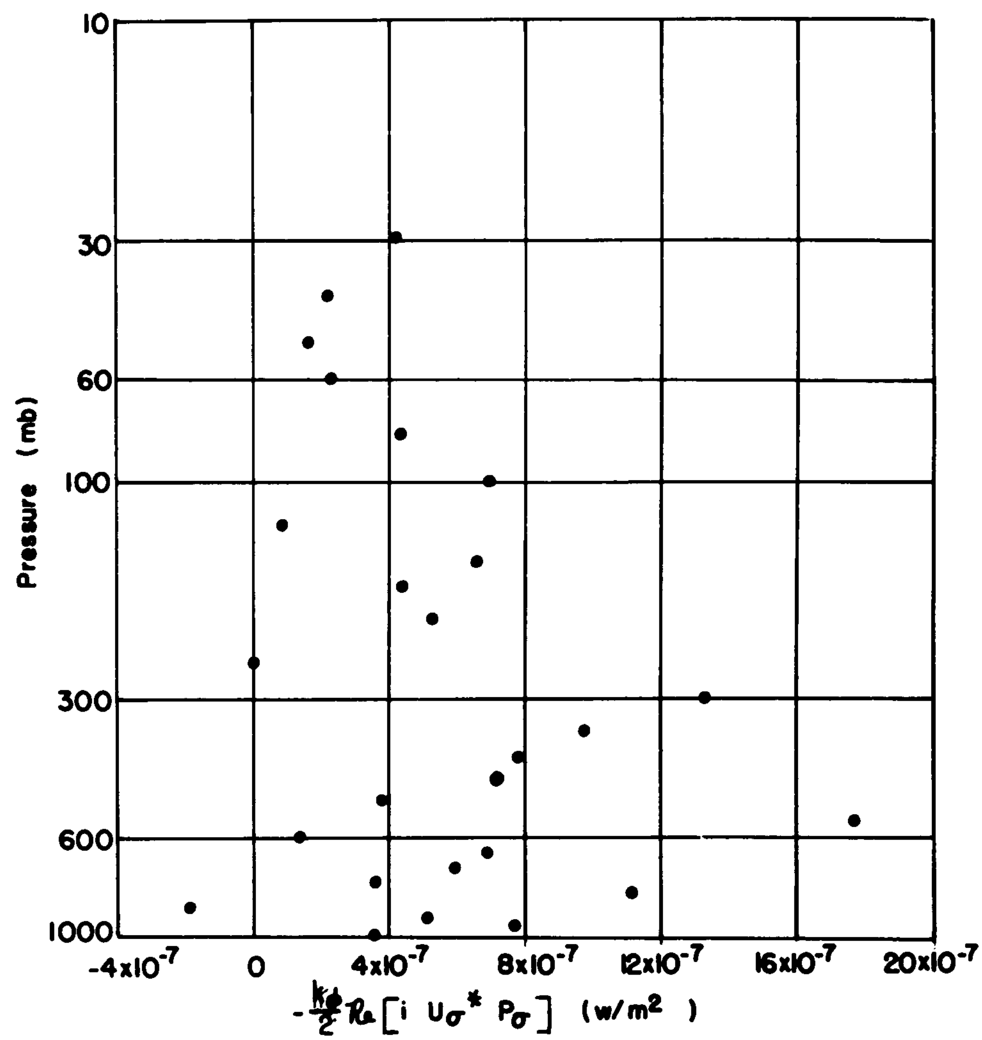

Figure 15. Generation of the zonal component of kinetic energy in the semidiurnal tide - Terciera.

or non-linear kinetic energy interaction may be responsible. Observations at other stations are needed to see whether this is a persistent phenomenon.

If it is assumed that only zonal wave number two is present, then from (10) and (11):

$$-\frac{1}{2} \mathrm{Re}\,[\,i\, u_r^{*} P_r\,] = \frac{1}{2} \mathrm{Re}\,[V_r^{*} P_r] \tag{19}$$

this appears to hold true, (Figures 13 and 14) except in the 500-200 mb region. On the assumption of a single zonal wave number two, the generation of zonal kinetic energy has been computed, (Figure 15) and shows a generation of zonal kinetic energy of the crder of 5×10^{-7} w./m.^{3} It should be noted that this is quite likely to be balanced by an equal loss of the meridional kinetic energy. Unfortunately, the latter cannot be computed without a knowledge of the meridional pressure variation.

It is not possible, on the basis of a single station, to decide whether the meridional energy flux is representative of a latitude circle, or will oscillate along it. If it is a representative flux, and if there are divergences of the order of $1/a$ of the flux itself, (a being the earth's radius) there may be energy divergences of the order of 2×10^{-3} w./m.^{2} of the earth's surface. This is not inappreciable compared to the 7×10^{-3} w./m.^{2} downward flux.

C. Observed Energy Transport - Fort Worth

A similar set of data for Fort Worth has been graciously provided in advance of publication, by Messrs. Harris and Finger, and Dr. Teweles. The

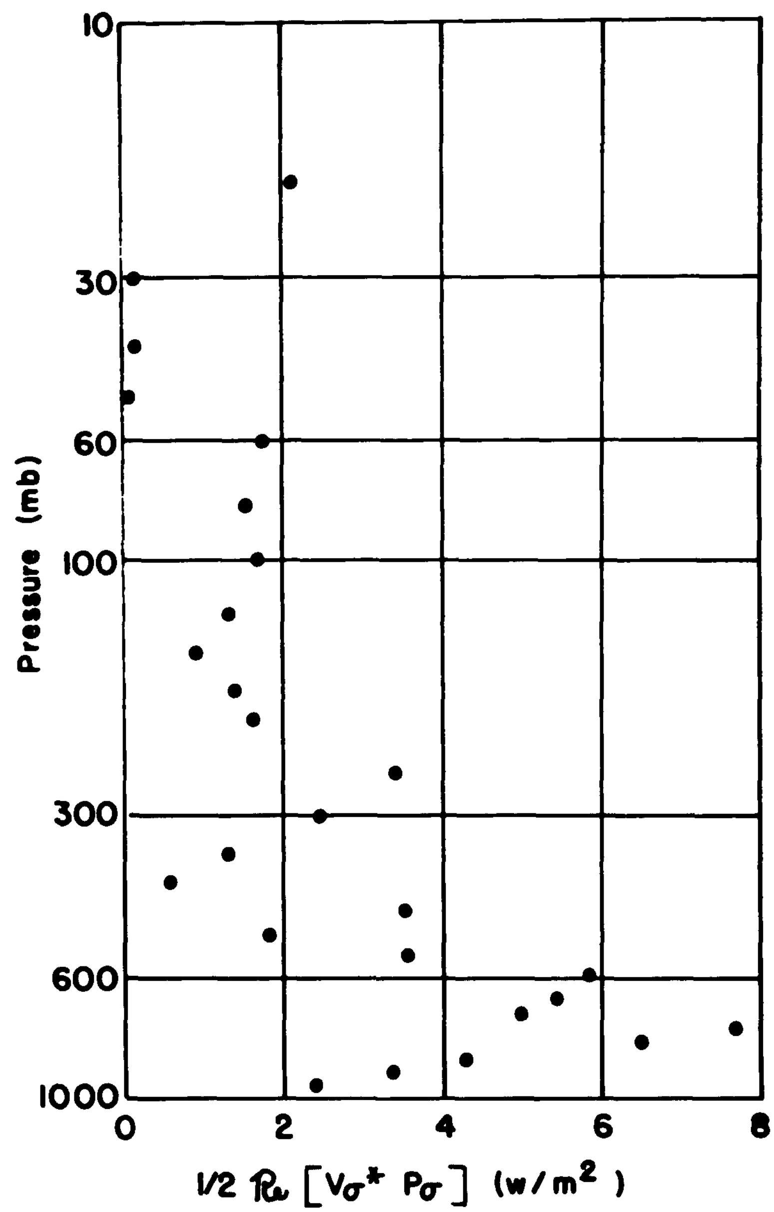

Figure 16. Meridional energy flux in the semidiurnal tide - Fort Worth.

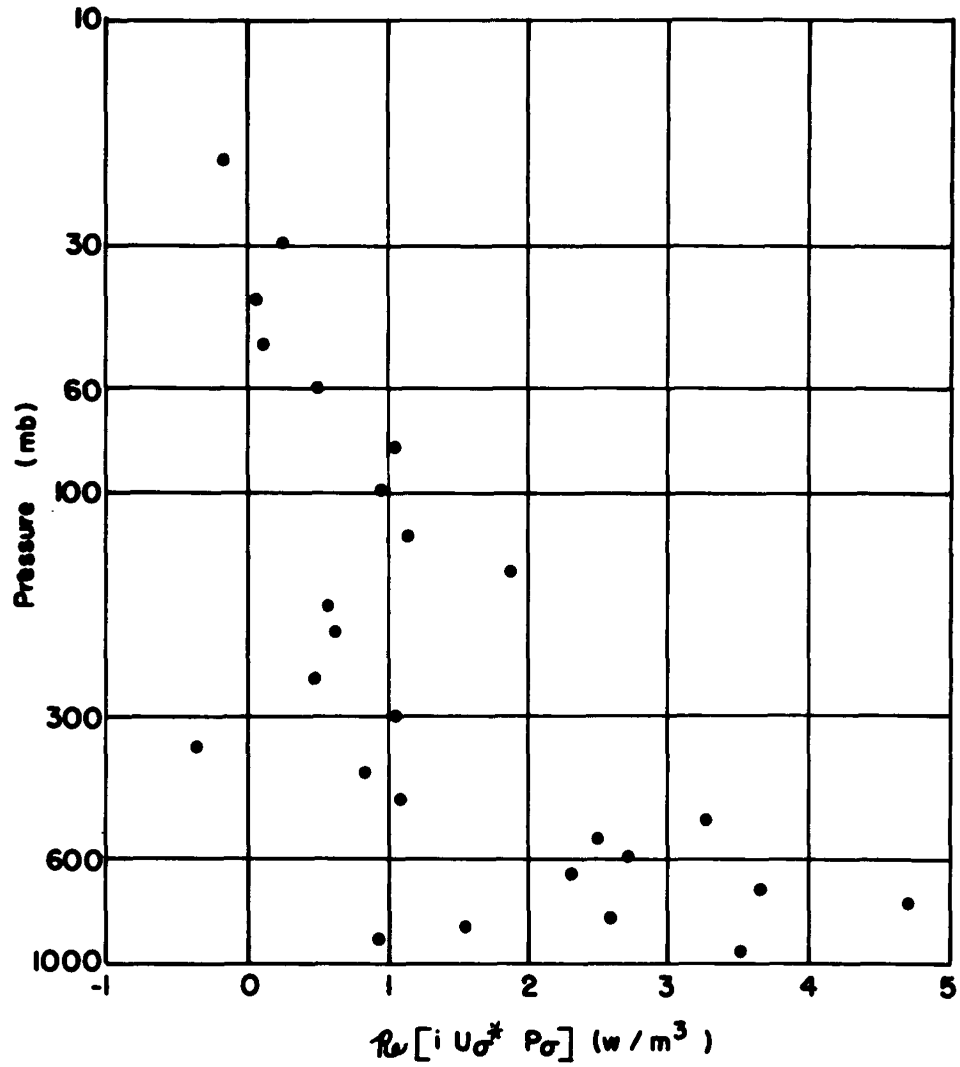

Figure 17. Rate of generation of zonal kinetic energy of the semidiurnal tide - Fort Worth.

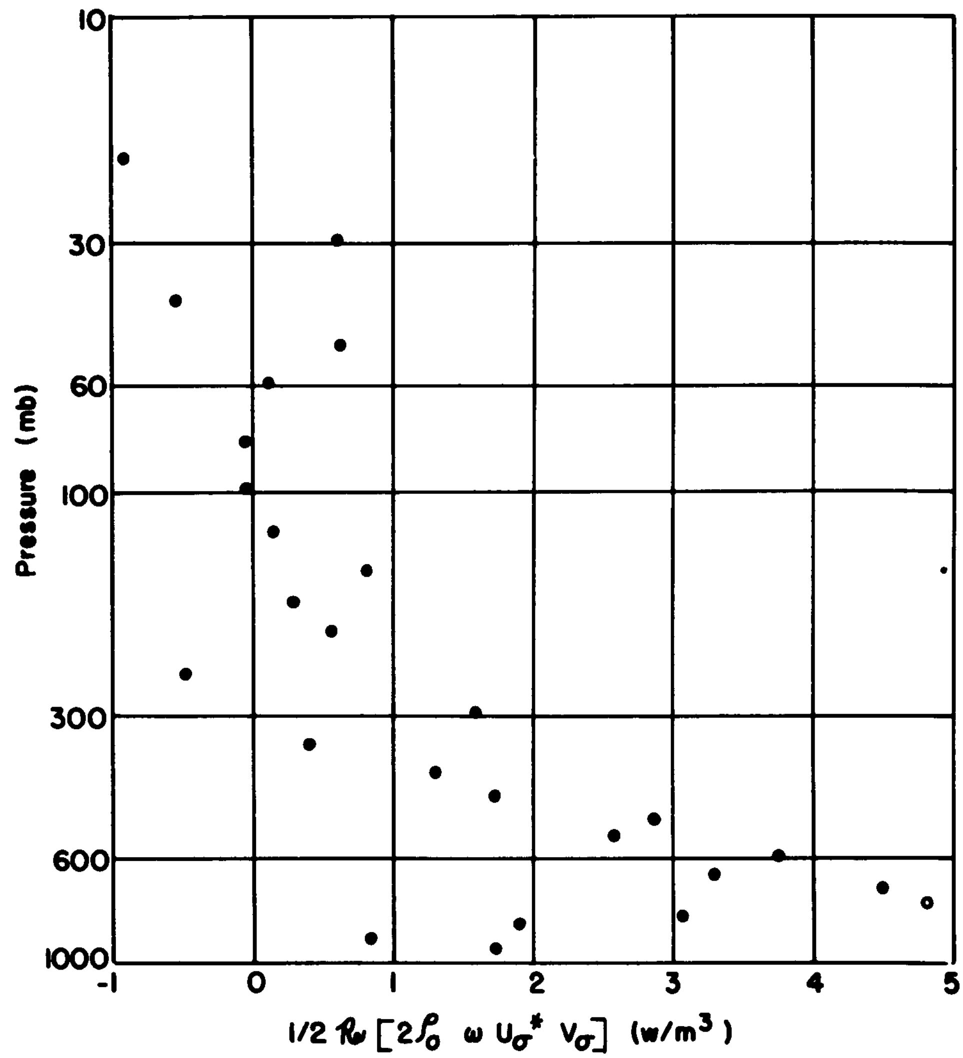

Figure 18. Rate of conversion of zonal to meridional kinetic energy of the semidiurnal tide - Fort Worth.

mean meridional energy energy flux, is shown in Figure 16. The flow is northward at all levels. The flux has a greater magnitude than at Terciera, not too surprising in view of the greater magnitudes of P_r and V_r. As the wind data at Fort Worth was originally recorded in smaller angular increments than the Terciera data, the tidal wind components have more accuracy, and the flux shows somewhat less scatter.

The generation of zonal kinetic energy from potential energy is shown in Figure 17, on the assumption of a wave of zonal number two. With the improved accuracy of the wind data, it is also possible to compute the conversion from zonal to meridional kinetic energy, as shown in Figure 18. This conversion largely balances the generation of zonal kinetic energy.

D. Meridional Transport of Momentum - Meteor Trail Observations

In the past decade, a new source of wind data for heights of 80 to 100 km has become available: radar observations of meteor trails, (Greenhow, 1959; Greenhow and Neufeld, 1955, 1956; Neufeld, 1958; Elford, 1953, 1959). These observations have been taken at Jodrell Bank, England, and Adelaide, Australia. They show very substantial tidal wind velocities of 10 to 50 meters per second. As the phases and amplitudes of the semidiurnal tide are available, it is possible to compute:

$$\frac{1}{2} \mathcal{R}_e \left(\rho_0 u_r^* v_r \right) \tag{20}$$

and hence to obtain the meridional transport of angular momentum by the tidal component of the winds.

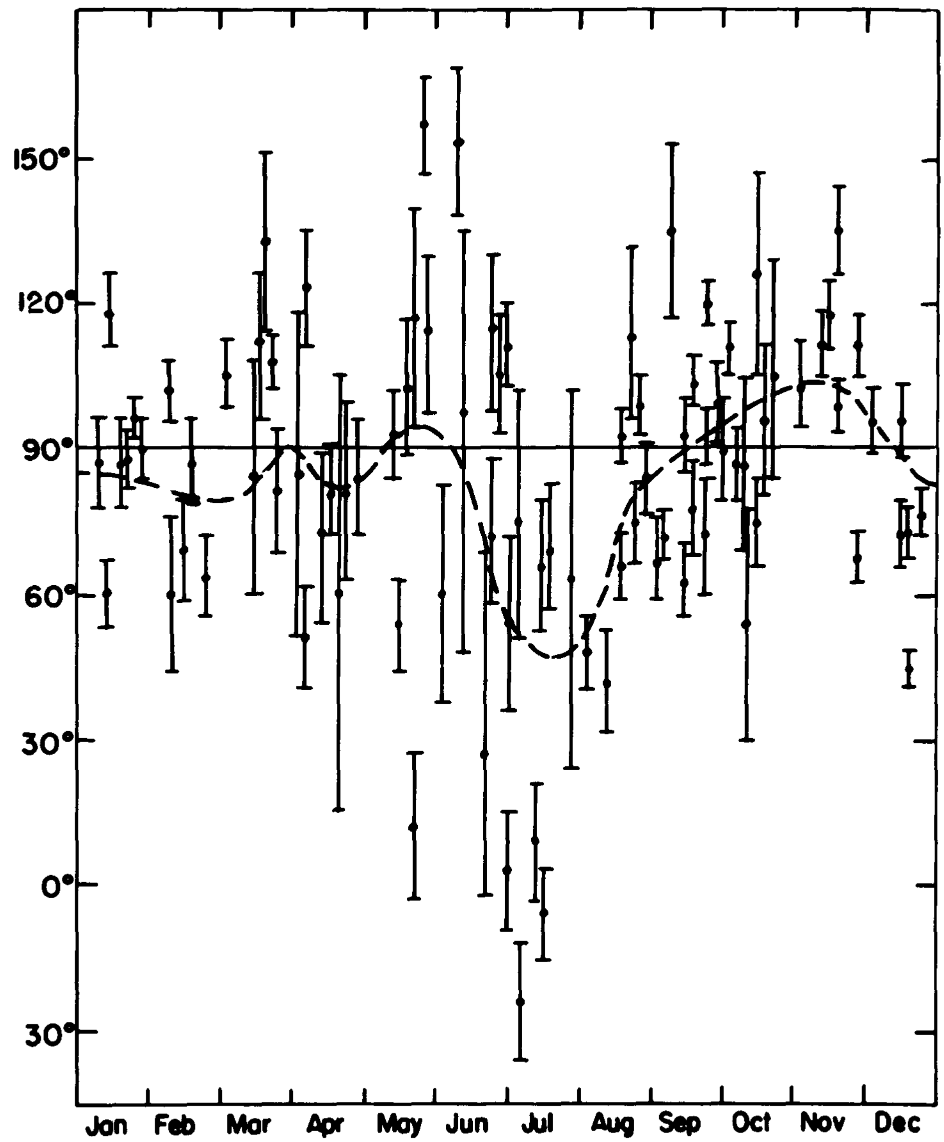

Figure 19. Phase angle between u and v semidiurnal wind components at a mean height of 92 kilometers - Jodrell Bank.

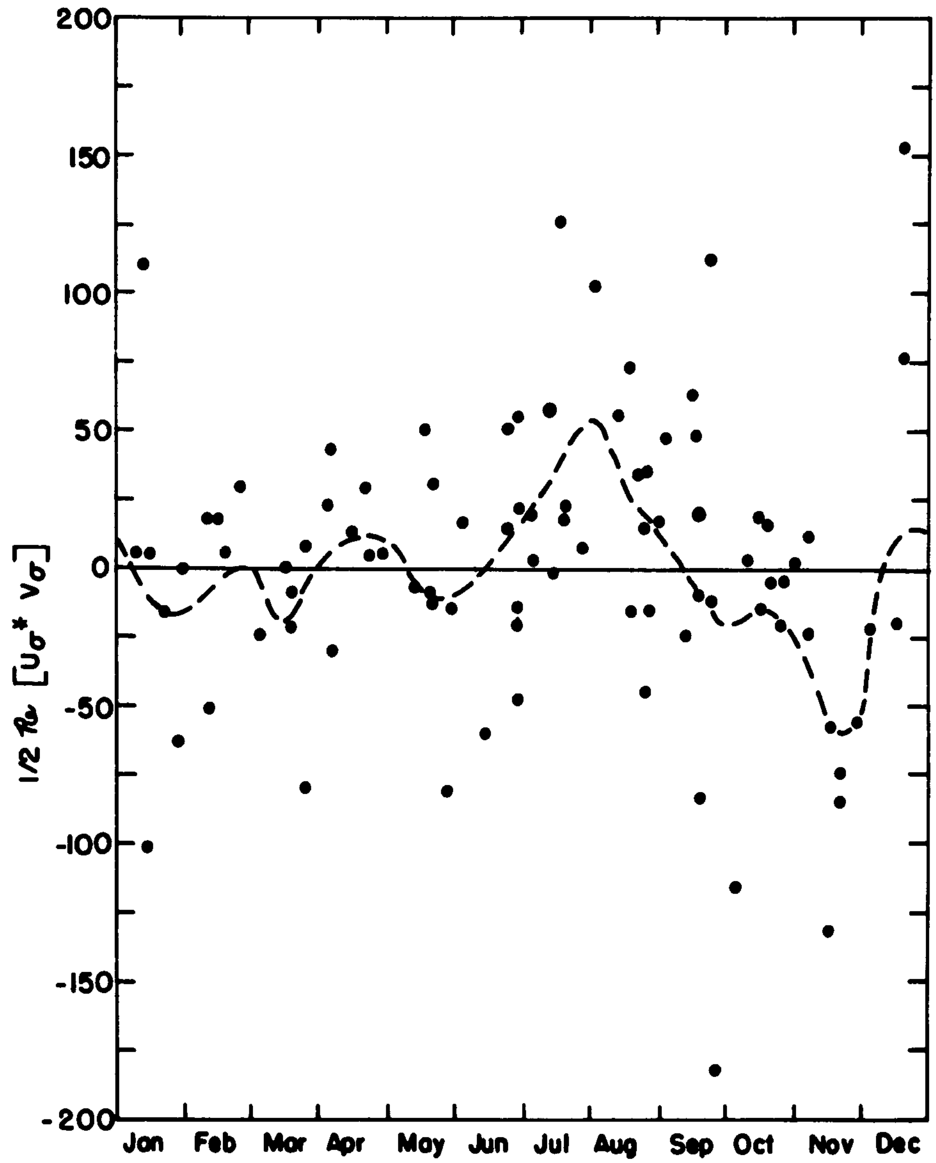

Figure 20. Meridional transport of zonal momentum at a mean height of 92 kilometers - Jodrell Bank.

At Jodrell Bank, the diurnal tide is considerably smaller than the semidiurnal tide, and will not be considered here. Neufeld, (1958) has presented data from 100 days of observations over a four year period from September, 1953 to August, 1957, averaged to refer to a mean height of 92 km. The relative phase angle between u_r and v_r is shown in Figure 19. The error flags are derived from Neufeld's estimate of the error. The dotted curve represents a smoothed average, obtained by averaging over a month, every half month. Figure 20 shows the values for (20) computed for each observation. The dotted line is a similarly smoothed average.

The data indicate a seasonal change in momentum transport. There is probably a northward transport in late winter and early spring, and a clear thought comparatively small transport in mid-summer. A southward transport occurs during October and November, corresponding to a sudden increase in the tidal magnitude and shift in phase. These transports are of a sufficiently great magnitude to be important in the general circulation, if they are maintained around a latitude circle, or to contribute to large standing waves if they are not. If:

$$\frac{1}{2} \mathrm{Re}\left[u_r^* v_r\right] = 24 \quad m^2/sec^2 \tag{21}$$

enough angular momentum is transported to increase the zonal wind by one meter per second per day everywhere north of Jodrell Bank.

Unfortunately, pressure data, (Greenhow, 1959) is not sufficiently accurate to permit computation of the meridional energy flux of the tide.

Density fluctuations are of the order of 7% of the total density, and adiabatic pressure variations would be of the order of $1/\gamma$ of this, or 5%. From the ARDC 1959 Model Atmosphere, $\rho_0 = 1.88 \times 10^{-6}$ kg./m.^{3}, and $P_0 = 9.07 \times 10^{-2}$ n./m.^{2} at 92 km. If $|\vec{V}_r| = 20$ m./sec, then:

$$\left| \tfrac{1}{2} \mathrm{Re} \left[\vec{V}_r \cdot P_r \right] \right| \lesssim 4.5 \times 10^{-2} \ \mathrm{w/m^2} \qquad (22)$$

With zonal momentum transports of this order, one might expect 10-20% of the energy flux to be in the meridional direction. As it requires a flux of 4×10^{-2} w./m.^{2} to heat the air at this level between Jodrell Bank and the pole by 1 degree per day, this flux is not an important part of the mesopause energy budget.

The data for Adelaide, (Elford, 1953, 1959) shows an increased semi-diurnal tide, and a greatly increased diurnal tide, now larger than its higher frequency counterpart. Both show velocities of 50 m./sec. While u_r and v_r often show strong time correlation for both tides, their phases vary quite erratically, and no seasonal trends have been observed.

CHAPTER IX. ADVECTION OF POTENTIAL ENERGY AND SENSIBLE HEAT

The transport of tidal energy has been considered in a Lagrangian frame of reference. It is also possible to consider transports by the tide in an Eulerian frame of reference, that is, by advection, across a surface fixed in space, (or at least fixed relative to the earth.) In this case tidal motions may also produce a mean flux of sensible heat:

$$\frac{1}{2}\,\mathrm{Re}\left[c_v\left(\rho_0 T_\sigma + \rho_\sigma T_0\right)\vec{V}_\sigma^{*}\right] \tag{1}$$

and of potential energy:

$$\frac{1}{2}\,\mathrm{Re}\left[\,\Phi\,\rho_\sigma \vec{V}_\sigma^{*}\right. \tag{2}$$

by advection. The significance of these energy transports can be made more clear by the following mental experiment.

Consider a volume of the atmosphere measuring one meter in height, one meter in zonal width, and on the average ten kilometers in meridional extent. Let the northern end of this volume move with the tide, so that no matter is advected across it, while the other surfaces remain fixed in space. The excursion of the moving surface is then given by, $V_\sigma / i\sigma$, of the order of one or two kilometers. For parameters comparable to the Terciera semidiurnal tide, in the troposphere:

$$\left|\,c_v\left(\rho_\sigma T_0 + \rho_0 T_\sigma\right)\right| \simeq 10^{2}\ \text{joule}/\text{m}^{3} \tag{3}$$

and $|\vec{V_r}| \simeq$ 0.2 m./sec., so that the mean advection of sensible heat through the southern end of the volume may be of the order of several watts per square meter, producing a total contribution of energy flow into the volume of a few watts.

The zonal advection of sensible heat per unit area is of the same order, but since the total surface is about 10^4 m^2 and zonal gradients are at most of the order of 10^{-7} or 10^{-6} of the flux itself, this contribution to mean acquisition of energy by the volume is negligible. Vertical advection will be discussed shortly.

The gravitational potential, Φ may be referred to any convenient level; it will be most convenient to take the bottom surface as the level at which is zero. Any other choice would produce different fluxes through each surface, but since the mean density does not increase, the sum of the differences would cancel. The mean advection of potential energy through the southern surface is:

$$\frac{1}{2} Re\left[\frac{\Phi}{2} \rho_r V_r\right] \lesssim 2 \times 10^{-4} \text{ watts} \tag{4}$$

which is negligible. The same type of argument applies to the lateral boundries as held for sensible heat, so that their net contribution may be ignored. There can be no advection of potential energy through the lower surface, where $\Phi = 0$. There is an instantaneous advection of:

$$-W_r (\rho_0 + \rho_r) g \quad w/m^2 \tag{5}$$

through the upper surface. Since the area of this surface varies considerably, a correlation of the flux and area can produce a mean acquisition of energy:

$$-\frac{1}{2}\,\mathcal{R}e\left[W_\sigma\,\rho_0\,g\,\frac{i V_\sigma^*}{\sigma}\right] \qquad (6)$$

within the volume. This can be significant, since:

$$\frac{|W_\sigma|\,\rho_0\,g\,|V_\sigma|}{2\sigma} \simeq 20 \quad W \qquad (7)$$

There is also energy transport by the term $\frac{1}{2}\mathcal{R}e[P_\sigma \vec{V}_\sigma^*]$ representing work done on the northern boundary, and the advection of energy of compression across the other, stationary surfaces. Accumulation of energy in the volume through the horizontal components of this flux are negligible, since gradients are small over the volume, and fluxes at opposite boundries largely balance.

Since:

$$P = \frac{\rho R T}{m} \qquad (8)$$

the vertical transports of sensible heat and energy of compression may be combined into a single term, with an instantaneous value of:

$$W_\sigma\left[\frac{C_v + R/m}{R/m}\right]P_0 = \frac{\gamma}{\gamma-1}\,W_\sigma\,P_0 \qquad (9)$$

The difference between fluxes at the two surfaces one meter apart lead to a net accumulation of energy per unit area from this term of:

$$-\frac{W_\sigma \gamma}{(\gamma-1)}\frac{\partial p_0}{\partial z} = \frac{\gamma}{\gamma-1} W_\sigma g \rho_0 \tag{10}$$

A mean accumulation of energy in the volume again results from the correlation of flux and area, and is:

$$\frac{\gamma}{2(\gamma-1)} \mathcal{Re}\left[W_\sigma \rho_0 g \frac{i V_\sigma^*}{\sigma} \right] \qquad \text{WATTS} \tag{11}$$

With a steady-state tidal system, there can be no mean accumulation of energy, and the advections of sensible heat, compressive energy, and potential energy must either balance each other, or be balanced by a non-adiabatic source or sink within the volume. The results of Chapter V indicate that, for the primary tide, viscosity is not an important energy sink. Insolational heating can provide such a source. The tidal heating rate per unit volume is 2×10^{-3} w./m.^{3} Since the volume changes, a correlation between the heating rate and the volume can produce a net energy acquisition, of the form:

$$\frac{1}{2} \mathcal{Re}\left[J_\sigma \rho_0 \frac{i V_\sigma^*}{\sigma} \right] \qquad \text{watts} \tag{12}$$

Equating this to the advective terms:

$$\frac{1}{2} \mathcal{Re}\left[J_\sigma \rho_0 \frac{i V_\sigma}{\sigma} \right] + \frac{1}{2} \mathcal{Re}\left[\frac{\gamma W_\sigma \rho_0 g}{(\gamma-1)} \frac{i V_\sigma^*}{\sigma} \right] + \frac{1}{2} \mathcal{Re}\left[C_v (\rho_\sigma T_0 + \rho_0 T_\sigma) V_\sigma^* \right] = 0 \tag{13}$$

or again making use of (8):

$$\frac{1}{2}\,\mathcal{R}e\left[P_\sigma V_\sigma^*\right] = \frac{(\gamma-1)}{2}\,\mathcal{R}e\left[c_v\left(\rho_\sigma T_0 + \rho_0 T_\sigma\right)V_\sigma^*\right]$$

$$= \frac{-1}{2}\,\mathcal{R}e\left\{\left[(\gamma-1)J_\sigma + \gamma W_\sigma g\right]\frac{i V_\sigma^* \rho_0}{\sigma}\right\} \qquad (14)$$

The meridional flux of tidal available potential energy is thus related to vertical advection processes and to the insolational heating. It may simply be the result of the phase relationships between various parameters necessary to produce vertical energy transport.

The above arguments were based on the assumption that only motions with frequency σ were present. One must consider whether energy convergence brought about by aperiodic motions or motions with other frequencies should be included in the volume mean energy balance. The choice of a mean meridional length of ten kilometers for this volume was arbitrary, and was made to simplify the problem conceptually. This length does not enter into the final results.

One might just as easily define the mean length as zero kilometers; in this case the volume, lateral length, and lateral areas must be considered to be negative half the time. A positive energy flux across a negative surface then corresponds to a negative addition of energy. If a proper regard for signs is maintained, the same results as above are obtained. There must be no mean accumulation of energy in the volume, even though its mean value is zero.

If the mean volume is zero, however, then a mean convergence of energy brought about by aperiodic motion or motion of other frequencies cannot contribute to the mean energy acquisition by the volume. They may contribute instantaneously, but their effect over time is averaged out.

CHAPTER X. NONLINEAR INTERACTION OF THE TIDES WITH ROSSBY WAVES

A. Introduction

Non-linear terms such as $(\vec{V}\cdot\nabla)\vec{V}$ have usually been neglected in the tidal equations, as being of small order. If the only motions present were those of the tides, this would be so, at least to the mesosphere, where tidal fluctuations become comparatively large. However, if the major atmospheric waves are considered, it is not a priori obvious that they do not contribute to equations of tidal frequency.

For example, consider the term $u_\sigma \frac{\partial u_a}{\partial \xi}$ where u_a is the non-tidal wind. Gradients of winds in the atmosphere are often 10^{-5} sec^{-1}, and may reach 10^{-4} sec^{-1}. This term must be compared with $i\sigma u_\sigma$, and $\sigma \simeq 1.5 \times 10^{-4}$ sec^{-1}. The generation of secondary waves through interaction between tidal and Rossby waves is thus a definite possibility.

In the following discussion, a mathematical derivation of the generation of gravity waves is set up, for a rotating planar geometry. The first case is applied, with simplifying assumptions, to a model atmosphere in order to obtain estimates of the energy transport that might be expected. It is assumed for tractibility that all waves have a definite frequency, and time-invariant amplitude.

B. Mathematical Development

First consider a flat planar geometry with an angular velocity $\vec{\omega}$, directed normally. It shall be assumed that the atmosphere has a barotropic

state, denoted by the subscript $_o$, and that superimposed on this state are waves, whose parameters are of small order. These waves may be of various types and frequencies, but it is assumed that they are steady-state and that all equations may be taken as Fourier transformed in time; all variables having the time dependence σ will be denoted by the subscript $_\sigma$. A product of two terms bracketed by $\{\quad\}_\sigma$ will denote the Fourier transform of all products having the resulting frequency σ , except for interactions involving the barotropic state.

Under these restrictions, the equations of motion become:

$$i\sigma u_\sigma - 2\omega V_\sigma = -\frac{1}{\rho_o}\frac{\partial P_\sigma}{\partial \xi} + \frac{1}{\rho_o^2}\left\{\rho\frac{\partial P}{\partial \xi}\right\}_\sigma - \left\{\vec{V}\cdot\nabla u\right\}_\sigma$$

$$i\sigma V_\sigma + 2\omega u_\sigma = -\frac{1}{\rho_o}\frac{\partial P_\sigma}{\partial \eta} + \frac{1}{\rho_o^2}\left\{\rho\frac{\partial P}{\partial \eta}\right\}_\sigma - \left\{\vec{V}\cdot\nabla V\right\}_\sigma \tag{1}$$

Here ξ and η are the horizontal coordinates, along which the velocities u and V flow, respectively. When equations (1) are solved for u_σ and V_σ :

$$u_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\left[\frac{i\sigma}{\rho_o}\frac{\partial P_\sigma}{\partial \xi} + \frac{2\omega}{\rho_o}\frac{\partial P_\sigma}{\partial \eta} + i\sigma\left\{\vec{V}\cdot\nabla u\right\}_\sigma \right.$$
$$\left. + 2\omega\left\{\vec{V}\cdot\nabla V\right\}_\sigma - \frac{i\sigma}{\rho_o^2}\left\{\rho\frac{\partial P}{\partial \xi}\right\}_\sigma - \frac{2\omega}{\rho_o^2}\left\{\rho\frac{\partial P}{\partial \eta}\right\}_\sigma\right] \tag{2}$$

and:

$$v_\sigma = \frac{1}{(\sigma^2 - 4\omega^2)}\Big[\frac{i\sigma}{\rho_0}\frac{\partial P_\sigma}{\partial \eta} - \frac{2\omega}{\rho_0}\frac{\partial P_\sigma}{\partial \xi} + i\sigma\{\vec{V}\cdot\nabla v\}_\sigma$$

$$+ 2\omega\{\vec{V}\cdot\nabla u\}_\sigma - \frac{i\sigma}{\rho_0^2}\{\rho\frac{\partial P}{\partial \eta}\}_\sigma + \frac{2\omega}{\rho_0^2}\{\rho\frac{\partial P}{\partial \xi}\}_\sigma\Big] \tag{3}$$

The velocity divergence, χ_σ is given by:

$$\chi_\sigma = \frac{\partial w_\sigma}{\partial z} + \frac{\partial u_\sigma}{\partial \xi} + \frac{\partial v_\sigma}{\partial \eta} \tag{4}$$

If also:

$$\mathcal{Q} \equiv \frac{1}{(\sigma^2 - 4\omega^2)}\Big[\Big(i\sigma\frac{\partial}{\partial \xi} + 2\omega\frac{\partial}{\partial \eta}\Big)\Big(\{\vec{V}\cdot\nabla u\}_\sigma - \frac{1}{\rho_0^2}\{\rho\frac{\partial P}{\partial \xi}\}_\sigma\Big)$$

$$-\Big(i\sigma\frac{\partial}{\partial \eta} + 2\omega\frac{\partial}{\partial \xi}\Big)\Big(\{\vec{V}\cdot\nabla v\}_\sigma - \frac{1}{\rho_0^2}\{\rho\frac{\partial P}{\partial \eta}\}_\sigma\Big)\Big] \tag{5}$$

and:

$$F \equiv \frac{a^2}{(\sigma^2/4\omega^2 - 1)}\Big(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\Big) \tag{6}$$

where a is a characteristic distance, then equation (4) may be written:

$$\chi_\sigma = \frac{\partial w_\sigma}{\partial z} + \frac{i\sigma}{4a^2\omega^2}F\Big(\frac{P_\sigma}{\rho_0}\Big) + \mathcal{Q}_\sigma \tag{7}$$

The hydrostatic approximation is made, so that $\frac{D w_\sigma}{D t} = 0$.
The vertical equation of motion then becomes:

$$\frac{\partial p_\sigma}{\partial z} = -g \rho_\sigma \tag{8}$$

The equation of continuity is:

$$i\sigma \rho_\sigma + w_\sigma \frac{\partial \rho_0}{\partial z} + \{\vec{v} \cdot \nabla \rho\}_\sigma + \{\rho \chi\}_\sigma + \rho_0 \chi_\sigma = 0 \tag{9}$$

Two more quantities are defined by:

$$\mathcal{H}_\sigma \equiv \frac{\{\vec{v} \cdot \nabla \rho\}_\sigma + \{\rho \chi\}_\sigma}{\rho_0} \tag{10}$$

$$\chi_\sigma' \equiv \chi_\sigma + \mathcal{H}_\sigma \tag{11}$$

so that the equation of continuity may be written:

$$i\sigma \rho_\sigma + \rho_0 \chi_\sigma' + w_\sigma \frac{\partial \rho_0}{\partial z} = 0 \tag{12}$$

If J_σ is the rate of non-adiabatic heating per unit time per unit mass, the first law of thermodynamics may be written in the form:

$$J_\sigma = c_v \frac{DT_\sigma}{Dt} + \left\{ p \frac{D}{Dt}\left(\frac{1}{\rho}\right) \right\}_\sigma + p_0 \frac{D}{Dt}\left(\frac{1}{\rho_\sigma}\right) \tag{13}$$

If this is combined with the equation of state for an ideal gas, one finds:

$$i\sigma p_\sigma + w_\sigma \frac{\partial p_0}{\partial z} = \gamma g H_0 \rho_0 \chi_\sigma + \gamma g \rho_0 \{H\chi\}_\sigma + \gamma g H_0 \{\rho\chi\}_\sigma + \gamma g \{H\rho\chi\}_\sigma - \{\vec{v}\cdot\nabla p\}_\sigma \qquad (14)$$

Let $\mathcal{J}_\sigma$ be defined by:

$$(\gamma-1)\rho_0 \mathcal{J}_\sigma \equiv -\gamma g H_0 \{\vec{v}\cdot\nabla\rho\}_\sigma + \gamma g \{H\rho\chi\}_\sigma + \gamma g \rho_0 \{H\chi\}_\sigma - \{\vec{v}\cdot\nabla p\}_\sigma \qquad (15)$$

where γ is the ratio of specific heats, and also let $J'_\sigma \equiv J_\sigma + \mathcal{J}_\sigma$. In the above equations, H_0 is the scale height of the atmosphere. Equation (14) then becomes:

$$i\sigma p_\sigma + w_\sigma \frac{\partial p_\sigma}{\partial z} = \gamma g H_0 \rho_0 \chi'_\sigma + (\gamma-1)\rho_0 J'_\sigma \qquad (16)$$

or:

$$i\sigma p_\sigma = w_\sigma g \rho_0 + \gamma g H_0 \rho_0 \chi'_\sigma + (\gamma-1)\rho_0 J'_\sigma \qquad (17)$$

If equation (11) is substituted into (7), one obtains:

$$\chi_\sigma' = \frac{\partial w_\sigma}{\partial z} + \frac{i\sigma}{4a^2\omega^2} F\left(\frac{p_\sigma}{p_0}\right) + \mathcal{Z}_\sigma + \mathcal{H}_\sigma \tag{18}$$

This is analogous to (II-25). The linear analysis may be followed almost directly from here.

Differentiate equation (17) with respect to z , to obtain:

$$\begin{aligned} i\sigma \frac{\partial p_\sigma}{\partial z} = {} & g\rho_0 \frac{\partial w_\sigma}{\partial z} - \frac{g w_\sigma \rho_0}{H_0}\left(1 + \frac{dH_0}{dz}\right) \\ & + \gamma g \rho_0 \chi_\sigma' - \gamma g \rho_0 H_0 \frac{\partial \chi_\sigma'}{\partial z} \\ & + (\gamma - 1)\frac{\partial}{\partial z}\left(\rho_0 J_\sigma'\right) \end{aligned} \tag{19}$$

Use has been made of the fact that:

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} = \frac{-1}{H_0}\left(1 + \frac{dH_0}{dz}\right) \tag{20}$$

The equation of continuity and the hydrostatic equation may be combined to obtain:

$$\begin{aligned} i\sigma \frac{\partial p_\sigma}{\partial z} &= -i\sigma g \rho_\sigma \\ &= \frac{w_\sigma}{g}\frac{\partial \rho_0}{\partial z} + g\rho_0 \chi_\sigma' \end{aligned} \tag{21}$$

Equations (19) and (21) may be combined to obtain:

$$0 = g\rho_0 \frac{\partial w_\sigma}{\partial z} + (\gamma+1) g \rho_0 \chi_\sigma' - \gamma g \rho_0 H_0 \frac{\partial \chi_\sigma'}{\partial z} + (\gamma-1)\frac{\partial}{\partial z}(\rho_0 J_\sigma') \tag{22}$$

If this equation and equation (11) are differentiated with respect to z, and $\frac{1}{\gamma}\frac{\partial^2 w_\sigma}{\partial z^2}$ eliminated between them, one obtains:

$$\frac{1}{\gamma}\frac{\partial \mathcal{X}_\sigma'}{\partial z} + \frac{g}{4a^2\omega^2} F\left[-\frac{i\sigma}{\gamma g}\frac{\partial}{\partial z}\left(\frac{p_\sigma}{p_0}\right)\right] - \frac{1}{\gamma}\frac{\partial}{\partial z}\left(\mathcal{Q}_\sigma + \mathcal{H}_\sigma\right)$$
$$= H_0\frac{\partial^2 \mathcal{X}_\sigma'}{\partial z^2} + \frac{dH_0}{dz}\frac{\partial \mathcal{X}_\sigma'}{\partial z} - \kappa\frac{\partial \mathcal{X}_\sigma'}{\partial z} - \frac{\kappa}{\rho_0 g}\frac{\partial^2}{\partial z^2}\left(\rho_0 J_\sigma'\right) \tag{23}$$
$$- \frac{\kappa}{\rho_0 g H_0}\left(1 + \frac{dH_0}{dz}\right)\frac{\partial}{\partial z}\left(\rho_0 J_\sigma'\right)$$

where $\kappa = \frac{\gamma - 1}{\gamma}$. But from (17) and (22):

$$-\frac{i\sigma}{\gamma g}\frac{\partial}{\partial z}\left(\frac{p_\sigma}{p_0}\right) = \left(\kappa + \frac{dH_0}{dz}\right)\mathcal{X}_\sigma' - \frac{\kappa}{g}\left(1 + \frac{\partial H_0}{\partial z}\right)J_\sigma' \tag{24}$$

$$H_0\frac{\partial^2 \mathcal{X}_\sigma'}{\partial z^2} + \left(\frac{dH_0}{dz} - 1\right)\frac{\partial \mathcal{X}_\sigma'}{\partial z} - \frac{\kappa}{g}\frac{\partial}{\partial z}\left[\frac{\partial J_\sigma'}{\partial z} - \left(1 + \frac{dH_0}{dz}\right)\frac{J_\sigma'}{H_0}\right]$$
$$+ \frac{1}{\gamma}\frac{\partial}{\partial z}\left(\mathcal{Q}_\sigma + \mathcal{H}_\sigma\right) - \frac{g}{4a^2\omega^2}F\left[\left(\kappa + \frac{dH_0}{dz}\right)\mathcal{X}_\sigma' \right. \tag{25}$$
$$\left. - \frac{\kappa}{g}\left(1 + \frac{dH_0}{dz}\right)J_\sigma'\right] = 0$$

Equation (25) may be solved by separation of variables. Let the variables $\mathcal{Q}_\sigma$, $\mathcal{H}_\sigma$, $\mathcal{X}_\sigma'$, and J_σ' be represented by series expansions in the eigenfunctions, Ψ_n of the operator F.

$$\begin{aligned}
\mathcal{Q}_\sigma &= \textstyle\sum_n \mathcal{Q}_n(z)\ \Psi_n(\xi,\eta) \\
\mathcal{H}_\sigma &= \textstyle\sum_n \mathcal{H}_n(z)\ \Psi_n(\xi,\eta) \\
\mathcal{X}_\sigma' &= \textstyle\sum_n \mathcal{X}_n'(z)\ \Psi_n(\xi,\eta) \\
J_\sigma' &= \textstyle\sum_n J_n'(z)\ \Psi_n(\xi,\eta)
\end{aligned} \tag{26}$$

When these are substituted into (25), with h_n taken as the separation coefficient:

$$F\Psi_n + \frac{4a^2\omega^2}{g h_n}\Psi_n = 0 \tag{27}$$

and:

$$H_o \frac{d^2 X_n'}{dz^2} + \left(\frac{dH_o}{dz} - 1\right)\frac{dX_n'}{dz} + \left(\kappa + \frac{dH_o}{dz}\right)\frac{X_n'}{h_n} \tag{28}$$

$$= \frac{\kappa}{g}\left\{\left(1 + \frac{dH_o}{dz}\right)\frac{J_n'}{H_o h_n} - \frac{d}{dz}\left[\left(1 + \frac{dH_o}{dz}\right)\frac{J_n'}{H_o}\right]\right\}$$

$$+ \frac{d^2 J_n'}{dz^2} - \frac{1}{\gamma}\frac{d}{dz}\left(\mathcal{G}_r + \mathcal{H}_n\right)$$

The value of the separation coefficient is determined by the eigen-solutions to equation (27). If $\mathcal{G}_n$, $\mathcal{H}_n$, J_n' , and h_n are known functions, then equation (28) formally gives a solution for X_n' and hence X_n . The other parameters of the nth mode may be solved in terms of X_n and the known parameters.

It is conventional to cast equation (28) into the form of a one-dimensional wave equation by two further changes of variable. Let:

$$x \equiv \int_0^z \frac{dz'}{H_o(z')} \tag{29}$$

and:

$$q_n e^{x/2} \equiv X_n' - \frac{\kappa J_n'}{g H_o} \tag{30}$$

In order to handle non-linear terms introduced in this derivation, a third quantity:

$$\mathfrak{J}_n e^{x/2} \equiv \frac{H_0}{\gamma} \frac{d}{dz}\left(\mathcal{Q}_n + \mathcal{H}_n\right) \tag{31}$$

is also defined. When (29), (30), and (31) are substituted into (28), the resulting equation is:

$$\frac{d^2 q_n}{dx^2} - \frac{1}{4}\left[1 - \frac{4}{h_n}\left(\kappa H_0 + \frac{dH_0}{dx}\right)\right] q_n = \frac{\kappa J_n'}{\gamma g h_n} - \mathfrak{J}_n \tag{32}$$

C. Simplification to an Isothermal Atmosphere

The following work will be devoted to isothermal model atmosphere. It will be convenient to develop the equations for this particular case. If H_0 is independent of z, equation (28) becomes:

$$H_0 \frac{d^2 X_n'}{dz^2} - \frac{dX_n'}{dz} + \frac{\kappa X_n'}{h_n} = \frac{\kappa}{g}\left\{\frac{J_n'}{H_0 h_n} - \frac{1}{H_0}\frac{dJ_n'}{dz} + \frac{d^2 J_n}{dz^2}\right\} - \frac{1}{\gamma}\frac{d}{dz}\left(\mathcal{Q}_n + \mathcal{H}_n\right) \tag{33}$$

or:

$$H_0 \frac{d^2 X_n}{dz^2} - \frac{dX_n}{dz} + \frac{\kappa X_n}{h_n} = -H_0 \frac{d^2 \mathcal{H}_n}{dz^2} + \frac{d\mathcal{H}_n}{dz} - \frac{\kappa}{h_n}\mathcal{H}_n - \frac{1}{\gamma}\frac{d}{dz}\left(\mathcal{Q}_n + \mathcal{H}_n\right) + \frac{\kappa}{g}\left\{\frac{J_n'}{H_0 h_n} - \frac{1}{H_0}\frac{dJ_n'}{dz} + \frac{d^2 J_n'}{dz^2}\right\} \tag{34}$$

In anticipation of subsequent results, it will be assumed that all variables on the right hand side of equation (34) have the vertical dependence $\exp\left[\left(\frac{1}{2}-\mu\right) z/H_0\right]$, where μ may be either real or complex. It will be shown in the simple models to be used here that $\mathcal{U}_n$, $\mathcal{V}_n$, and $\mathcal{J}_n$ are of this form. While it will not enter the following equations, Siebert indicates insolational heating may follow a similar law. On this assumption, equation (34) becomes:

$$H_0 \frac{d^2 X_n}{dz^2} - \frac{d X_n}{dz} + \frac{\kappa X_n}{h_n} = \frac{\mathcal{L}_n}{H_0} e^{\left(\frac{1}{2}-\mu\right) z/H_0} \tag{35}$$

where:

$$\begin{aligned} \mathcal{L}_n e^{\left(\frac{1}{2}-\mu\right) z/H_0} \equiv\ & \mathcal{V}_n\left[-\left(\tfrac{1}{2}-\mu\right)^2 + \left(\tfrac{1}{2}-\mu\right) - \frac{\kappa H_0}{h_n}\right] \\ & + \frac{\kappa J_n'}{g H_0}\left[\left(\tfrac{1}{2}-\mu\right)^2 - \left(\tfrac{1}{2}-\mu\right) + \frac{H_0}{h_n}\right] \\ & - \frac{1}{\gamma}\left(\tfrac{1}{2}-\mu\right)\left(\mathcal{U}_n + \mathcal{V}_n\right) \end{aligned} \tag{36}$$

Again, a simplification of form may be had by changes of variables:

$$x \equiv z/H_0 \tag{37}$$

$$y_n \equiv X_n e^{-x/2} \tag{38}$$

$$\lambda_n^2 = -\frac{1}{4}\left[1 - \frac{4\kappa H_0}{h_n}\right] \tag{39}$$

If these are substituted in (36), it becomes:

$$\frac{d^2 y_n}{d x^2} + \lambda_n^2 y_n = \mathcal{L}_n e^{-\mu x} \tag{40}$$

D. Application to an Elementary Model

The preceding analysis has been applied to a very simple model, in order to obtain an order of magnitude estimate of the energy losses to secondary tidal waves through interaction with Rossby waves. Details of the model are as follows.

1. The atmosphere is planar, rotating with angular velocity 0.5×10^{-4} sec^{-1}, and isothermal, with a scale height of 6 km. The gravitational acceleration is 9.8 m/sec^2. The basic wavelength in the ξ direction is 30,000 km, and all waves are uniform in the η direction. The ground is taken to be smooth.

2. There is an unchanging wave of the form:

$$V = 3.0\, e^{i k_1 \xi} \quad m/sec \tag{41}$$

and a wavelength of 7500 km. The amplitude is chosen to be compatible with a spectral analysis of large scale waves by Saltzman, (1956), and a similar analysis by Horn and Bryson, (1963). The wave does not change with height, and the pressure field is geostrophic.

3. There is a tidal wave, whose ξ velocity component is:

$$u_n = 0.2\, e^{i(\sigma x + k_2 \xi)}\, e^{-(\frac{1}{2} - \mu) z / H_0} \tag{42}$$

(Therefore, y_n varies as $e^{-\mu x}$). The horizontal wavelength is 15,000 km, and the frequency $\sigma = 1.5 \times 10^{-4}$ sec^{-1}. It is assumed that non-linear effects are small, and that first order values for other parameters of this wave derive from linear tidal theory. This theory and the given data then determines that:

$$\mu = 0.188 \tag{43}$$

In deriving the other tidal parameters, it is assumed that there is no heating function, and no restriction on vertical motion at the surface. In effect, the primary tide might be considered to be driven by undulations on the surface of the earth. (While this is highly artificial, the object is merely to get representative tidal motions in the atmosphere itself.

4. A secondary tidal wave of the form:

$$u = u_n(z)\, e^{i[\sigma x + (k_1 + k_2)\xi]} \tag{44}$$

is generated. χ_n for this wave is computed from the equations of the preceding sections, and the energy transmitted to great heihgts is calculated. It is assumed that the secondary, like the primary tidal wave, does not interact strongly the other waves, to form tertiary waves.

The vertical energy flux computed for this wave is 5×10^{-5} w./m.^{2} It is felt that this is a representative order of magnitude value for waves of these dimensions. There was no large cancellation of terms, and individual terms checked for other forms of Rossby wave were of the same magnitude as their counterparts in this model. All parameters, including μ , were chosen to be as representative as possible.

The upward flux of a single secondary wave is only of the order of 1% of the generation of tidal energy. In order to know whether non-linear interactions are a serious loss mechanism for the tide, it is necessary to answer to further questions, as to the energy flux when summed over all wave numbers, and as to the relative amounts of energy contributed to this flux by the primary tidal and Rossby waves.

For large wave numbers, equation (83) may be used to show that:

$$W_n \propto \frac{U_1^2(0)}{\lambda_n} \propto U_1^2(0)\, k_1 \qquad (45)$$

If $U_1^2(0)$ may be expressed in terms of k_1^{-b}, where b is some empirically or theoretically determined exponent, then one may make at least qualitative conclusions about the summation of flux over wave numbers. If $b < -2$, the summation converges. For wave numbers above 6 or so, Horn and Bryson, (1963) find b = -8/3 out to wave number 12. Ogura, (1958) found b = -7/3 over this spectral region. If this behavior continues to higher wave numbers, the secondary flux summation should converge reasonably rapidly. Unfortunately very little is known about atmospheric eddies with dimensions from 100 to 1000 km, at least insofar as their mean kinetic energies are concerned.

There is a second reason for believing that smaller eddies will not contribute greatly. The preceding calculations were based on the assumption that the Rossby waves do not change amplitude. This is not a particularly good assumption in any case, but becomes increasingly poor for the smaller

scales. The rate of energy transfer to a secondary wave depends among other things on the amplitude of that wave, and under transient circumstances, it will take that wave several cycles to build up to magnitudes comparable to steady-state values. It seems likely that energy transfer will be less than computed here under such circumstances.

Finally, one may note that the non-linear generation of energy in wave A occurs through the advection of a parameter, (say velocity) of wave B by wave C up the gradient of a corresponding parameter, (say momentum) of wave A. Saltzman, (1955). For example, consider the kinetic energy exchanges involving the u velocity components of three such waves. The rates on energy generation are proportional to:

$$
\begin{aligned}
&\mathrm{Re}\left[i k_A \; u_A \; u_B \; u_C \right] \\
&\mathrm{Re}\left[i k_B \; u_A \; u_B \; u_C \right] \\
&\mathrm{Re}\left[i k_C \; u_A \; u_B \; u_C \right] \\
&\quad k_1 + k_2 + k_3 = 0
\end{aligned}
\tag{46}
$$

If $k_A \ll k_B$, then energy exchanges occur primarily between waves B and C. When one considers the interaction between the primary tide, Rossby waves of short length, and secondary tides, the energy transfer will be primarily between the latter two.

Since it appears that secondary wave fluxes are small, and extract energy from the Rossby waves, they are not regarded as an important loss mechanism. It should be stressed, however, that the approximations used in this chapter are of a rather crude nature.

CHAPTER XI. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

A. Conclusions

The semidiurnal atmospheric tide has been found to transport available potential energy downward; at Terciera, in the Azores, this flux amounts to some 7×10^{-3} watts per square meter at or near the ground. The primary source of this energy appears to be insolational heating by water vapor in the troposphere, though a horizontal convergence of tidal energy may make appreciable contributions.

Three possible sinks for this energy have been considered. These are:

1. A tidal-terrain interaction.
2. Viscous damping near the ground.
3. Convective heating out of phase with the tidal temperature fluctuation.

On the basis of rather simplified theory, the latter two possibilities appear too small, and would be expected to produce effects not found in observational data. They are of a magnitude that might account for local perturbations in the semidiurnal tide. It has not been possible to make an accurate estimate of the tidal-terrain effect. To do so would involve treating several hundred coupled equations involving modes of oscillation expressed in complicated Hough's functions. A simplified analysis has led to an estimated energy flux of 0.5×10^{-3} watts per square meter or more, with no way of telling by what factor this is an underestimate. The tidal-terrain effect thus cannot be ruled out, though evidence for its importance is far from conclusive.

This effect has been obscured by the traditional approximation of a smooth earth surface as a boundary condition. a rough boundary acts to couple all possible modes of oscillation, so that the primary mode of tidal oscillation cannot be excited without also exciting a large number of secondary modes. Most of the latter are internal gravity waves, and transport energy upward. There can be no net energy transport through the ground, so that a corresponding downward flux of nergy must exist in the primary, or driven tidal mode.

The secondary waves have horizontal lengths from global scale down to the order of one hundred kilometers, and possibly smaller. Viscous damping in the troposphere is substantial for wavelengths shorter than roughly one thousand kilometers. Longer waves transport tidal energy to the upper mesosphere, where increased eddy viscosity damps them. The largest waves reach the lower thermosphere, where molecular viscosity and possibly hydromagnetic damping are effective. In these high regions, non-linear interactions between the waves may be important.

The observed data for the primary semidiurnal tide above the Azores and Fort Worth also show a meridional transport of energy and angular momentum. These phenomena are not pobbible with a single linear, inviscid mode of oscillation, but may reflect interaction between two or more modes. More data is needed before any conclusions can be reached about these transports.

A set of equations has been developed to treat non-linear interactions between waves whose frequencies and amplitudes remain constant. Applied to

a rather simple model these equations indicate that interactions between tidal waves and the large scale atmospheric eddies do not represent an important tidal energy loss in the troposphere.

B. Suggestions for Future Work

The hypotheses developed in this study have been confirmed only by limited observational data. There is a need for further analyses of the type carried out by Harris, Finger, and Teweles, (1962), for Terciera. New data can serve not only to measure the vertical energy flux but to confirm the hypothesis that the $\Theta^{2}_{2\,2}$ mode of oscillation is dominant in the stratospheric as well as the ground level semidiurnal tide. Additional data is even more important in examining meridional transports, since these measurements are made with considerably lower accuracy and may vary with longitude as well as height and latitude. A more complex analysis of the meridional energy flux is necessary to assess its contribution to $\mathcal{R}_e(\nabla\cdot\vec{V}_\sigma^{*}\ P_\sigma)$.

Unfortunately, more than four daily observations are needed to obtain data on the semidiurnal tide. The technique that has been used is to choose a station taking four daily observations, but at different times in different years. The number of stations filling this requirement is quite limited.

It is possible that the amplitudes and phases of the $\Theta^{2}_{2\,2}$, (main migrating semidiurnal) and $\Theta^{0}_{2\,2}$, (main standing semidiurnal) modes of oscillation may be related through the tidal-terrain theory. Until now, it

has been assumed the latter arises out of differences in heating over land and sea, (Siebert, 1960).

A further expansion of meteor trail observations would add greatly to the knowledge of high level tides, as well as providing information about the general circulation at these levels. At present essentially nothing is known about the horizontal structure of either in the upper mesosphere.

The gap in information between 30 and 80 kilometers will be harder to fill. Rocket soundings have begun to provide wind data in this region; their number will have to be increased considerably before time harmonic analyses can be made with any accuracy, especially since the tides show marked seasonal changes in the upper atmosphere.

In short, the greatest need in tidal investigation is for more data at all levels above the ground.

The greatest challenge to the theoretician is to be found in the tides of the upper atmosphere. At the mesopause they are of a magnitude to be a major part of the atmospheric circulation. They may play important roles in the production, maintainance, and stability of the general circulation. Linear theory is no longer valid for these waves. Techniques such as were developed in Chapter IX may be applicable, or it may prove more advantageous to abandon the primitive equations, as has been done in dynamic meteorology.

At still higher levels, the coupling of hydrodynamic and hydromagnetic waves can be the basis of a challenging career. The analysis of hydromagnetic damping has been done in a very crude manner, and the possibility that hydromagnetic waves are generated has been completely ignored. If tidal theory is to be extended to the upper ionosphere, these problems must be faced.

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This study is but one in a long line of works on tidal theory, and rests firmly on the foundations laid by earlier workers. Thanks are thus due to all who have contributed to the understanding of this phenomenon.

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